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INVESTIGATION OF NONIDEAL PLASMA PROPERTIES.(U)
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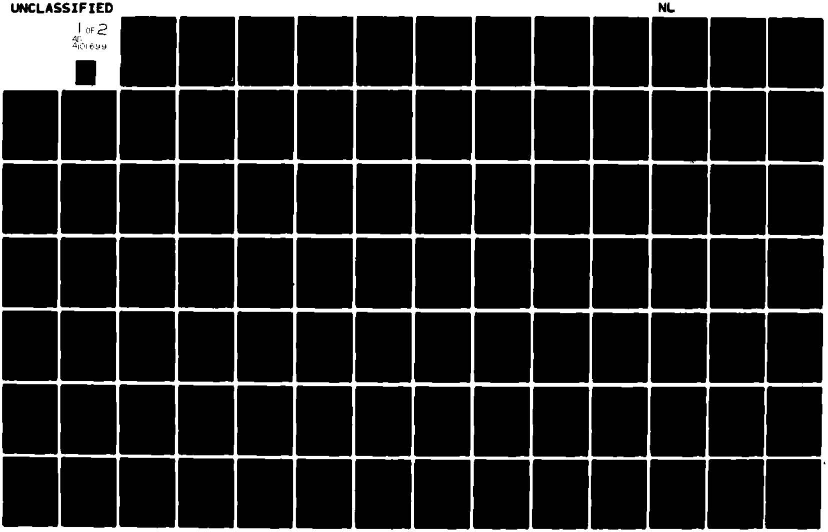

ANNUAL REPORT

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INVESTIGATION OF NONIDEAL PLASMA PROPERTIES

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C O N T E N T S

I. INTRODUCTION

This annual report contains investigations on the properties of nonideal plasmas, which were carried through in the period from 1 November 1978 to 1 November 1979, under ONR Contract N00014-79-0073. Progress was made in the evaluation of the electrical conductivity and the thermodynamic functions of nonideal plasmas. Research which was published in this period is not included.

Chapter II gives a dimensional analysis of the possible electrical conductivities of nonideal classical and quantum plasmas, with and without thermal effects. Quantum effects are considered since the electrons are noticeably degenerate for densities $n \geq 10^{21}\,cm^{-3}$ and temperatures $T \sim 10^{4\,o}K$. In strongly nonideal plasmas, $\gamma = Ze^2 n^{1/3}/KT >> 1$, the thermal energy is negligible compared with the energy of the Coulomb microfields, i.e. such plasmas behave like zero-temperature systems.

Chapter III presents a kinetic theory of the electrical conductivity of nonideal plasmas, when the electrons have i) a Maxwell and ii) a Fermi distribution of velocities. This work is based on the quantum mechanical scattering cross section for an effective, shielded Coulomb potential, which is applicable to intermediate nonideal conditions, $0.1 \lesssim \gamma \lesssim 10$. For moderate degeneracy, $n \lesssim \tilde{n}$, the conductivity depends on a Coulomb logarithm, but not for complete degeneracy, $n >> \tilde{n}$, where $\tilde{n} = 2(2\pi mKT/h^2)^{3/2} = 4.828 \times 10^{15} T^{3/2}$.

Chapter IV is concerned with a statistical theory of the free energy of nonideal gaseous $(0 < \gamma < 1)$ and quasi-liquid $(1 < \gamma < \gamma_c)$ plasmas, where γ_c is the critical interaction parameter of solid plasma. The physical model used considers short and long range Coulomb interactions through quasi-lattice interactions and collective electron and ion waves. The degeneracy effects on the free energy are discussed for large interaction parameters γ.

Chapter V derives the Hamilton function and the canonical field variables for a many-component plasma continuum with longitudinal Coulomb field interactions. By Fourier analyzing the canonical fields for random fluctuations the Hamilton function can be used to calculate the free interaction energy of nonideal plasmas. This method has, however, not yet lead to concrete statistical applications because of the mathematical difficulties associated with the evaluation of the complex integrals in the multi-dimensional phase space of the Fourier amplitudes of the canonical fields.

Chapter VI gives a simple application of the theoretical approach in Chapter V concerned with the evaluation of the distribution function of the velocity fluctuations in a neutral one-component gas. The theoretical distribution function is shown to be in agreement with the observations in turbulent gases.

Chapter VII is an appendix, in which an unrelated subject is discussed, namely the propagation of stress relaxation waves and their use for signal propagation and system detection in water.

The main results obtained are summarized in the abstracts of Chapters II - VII. Other research concerned with calculations of the electrical conductivity 1) using quantum-field theoretical methods of solid state physics and 2) considering the effects of the fluctuations of the Coulomb microfields on the current transport could not be written up in time, and will be communicated separetely. The same holds for the Ph.D. thesis of Mr. A.H. Khalfaoui on nonideal plasmas.

II. DIMENSIONAL ANALYSIS OF ELECTRICAL CONDUCTIVITY OF NONIDEAL CLASSICAL AND QUANTUM PLASMAS

Abstract

By means of dimensional analysis, novel formulas for the electrical conductivity of nonideal i) classical and ii) quantum plasmas are derived based on the axioms of Dupré. In the general case of a nonideal plasma with partially degenerate electrons, the conductivity is of the form $\sigma = C_\sigma \gamma_T^{-A} \gamma_Q^{-B} e n^{1/2}/m^{1/2}$, where C_σ is a dimensionless constant, A and B are powers, and $\gamma_T = e^2 n^{1/3}/KT$ and $\gamma_Q = e^2 n^{1/3}/(\hbar^2/mn^{-2/3})$ are the reduced (Z=1) nonideality parameters of the classical and quantum plasma, respectively (e, m, n are the charge, mass, and density of the electrons, KT is the thermal energy, and $\hbar$ is Planck's constant). The known conductivities are obtained as special cases of this conductivity formula, e.g., the conductivity $\sigma = C_\sigma (KT)^{3/2}/m^{1/2}e^2$ of the ideal, classical plasma, or the conductivity $\sigma = C_\sigma (me^6/\hbar^3)n^{1/3}/KT$ of the high-temperature metal.

INTRODUCTION

In recent years, numerous measurements of the electrical conductivity of non-ideal plasmas have been reported,[1-8] which were produced by oven heating [9] (medium pressures) and shock wave heating [10] (high pressures) of alkali vapors and noble gases, with typical pressures ranging from 10^0 to 10^5 bars. In spite of the availability of an approximate kinetic equation for nonideal plasmas,[11] which considers spatial and temporal correlations in the collision operator, satisfactory theoretical explanations of the experimental conductivity data on nonideal plasmas are missing to date. The degree of the nonideality of a fully ionized plasma is measured in terms of the (dimensionless) interaction parameter γ, which represents the ratio of average Coulomb interaction ($Ze^2n^{1/3}$) and thermal (KT) energies (n = electron density, Z = ion charge number, e = elementary charge),

$$\gamma = Ze^2n^{1/3}KT \quad .$$

In cgs-unit, $\gamma = 1.670 \times 10^{-3} Zn^{1/3}T^{-1}$. The conductivity theory of ideal, fully ionized plasmas[12] agrees with the experimental data only if $\gamma \ll 1$. For moderately, $0.1 < \gamma \leq 1$, nonideal plasmas, the ideal conductivity theory yields much too large conductivity values[12] $\sigma \sim (KT)^{3/2}/m^{1/2}e^2 Z \ell n \Lambda$, where $\Lambda = [1+(D/p_o)^2]^{1/2} = D/p_o$ for $D \gg p_o$, $p_o = Ze^2/2KT$ is the impact parameter for 90^o deflections, and D is the maximum impact parameter. The electric shielding length of Debye is related to the interaction parameter γ by

$$D = [Z/4\pi(1+Z)]^{1/2}\gamma^{-1/2}n^{-1/3} \quad .$$

The ideal conductivity theory breaks down at higher electron densities because the Debye radius D loses its physical meaning as an electric shielding length and upper impact parameter when the number of electrons in the Debye sphere, $N_D = 4\pi D^3 n/3$, is no longer large compared with one. N_D and γ are related by

$$N_D = (4\pi/3)[Z/4\pi(1+Z)]^{3/2}\,\gamma^{-3/2}.$$

For typical nonideal conditions, $n > 10^{20}\,cm^{-3}$ and $T = 10^{4\,o}K$, the Debye radius is $D < 10^{-8}$ cm, i.e., is smaller than the atomic diameter, which shows that the electric shielding concept is not applicable to proper nonideal plasmas, $\gamma \gtrsim 1$. Another reason for the inapplicability of the ideal conductivity theory to proper nonideal plasmas is its assumption of successive, small binary interactions, whereas in reality a conduction electron experiences many-body interactions for $\gamma \gtrsim 1$.

In the following, we apply dimensional theory to the derivation of new formulas for the electrical conductivity of (non-relativistic) ideal and nonideal, classical and quantum plasmas. In the most general case of an electron plasma, the electrical conductivity σ is a power function of the characteristic plasma parameters, $\sigma \sim e^p m^q \hbar^r (KT)^s n^t$ ($\hbar$ is Planck's constant). As special cases, the conductivity formulas for the ideal, classical, fully ionized plasma and the partially degenerate, solid metal are obtained. The derived formulas for nonideal classical and quantum plasmas indicate the dependence of σ on γ, which can be compared with the experimental observations.

THEORETICAL FOUNDATIONS

In a system of reference in which magnetic fields are absent, a linear electric current response $\vec{j} = \sigma \vec{E}$ exists, provided that the generating electric field $\vec{E}$ is sufficiently weak. For any gaseous, liquid, or solid medium, the electrical conductivity $\sigma = |\vec{j}|/|\vec{E}|$ is given by

$$\sigma = (ne^2/m)\tau \tag{1}$$

where e is the charge, m is the mass, n is the density, and τ is the (average) momentum relaxation time of the current carriers. Because of the large ion mass $m_i >> m$, the main current carriers in a plasma are the electrons. Eq. (1) holds for any perturbed Maxwell or Fermi distribution of the electrons.

Dimensional analysis is based on the axioms of Dupre.[13] By axiom 1), absolute numerical equality of quantities a and b may exist only when the quantities are similar qualitatively. That is, a general relation may exist between two quantities a and b only when the two quantities have the same dimension. By axiom 2), the ratio of the magnitudes of two like quantities a and b is independent of the units used in their measurement, provided that the same units are used for evaluating each.

In general, any measurable quantity σ (the secondary quantity) can be expressed in terms of those appropriate quantities a_i, $i = 1,2,\ldots M$ (the primary quantities), which affect the magnitude of σ. The general relationship between the magnitude of the secondary quantity σ and the magnitudes of the primary quantities a_i is a function of the M arguments of the form

$$\sigma = f(a_1, a_2, a_3, \ldots a_M) \quad . \tag{2}$$

Application of the axioms 1) and 2) to Eq. (2) demonstrates that the functional relation $f(a_i)$ is the power function[14)]

$$\sigma = C_\sigma a_1^{N_1} a_2^{N_2} a_3^{N_3} \dots a_M^{N_M} \quad . \tag{3}$$

C_σ is a dimensionless coefficient, which depends on the nature of the physical quantity σ, and can only be determined by means of a detailed physical model. In many cases, the order-of-magnitude of C_σ is one, $C_\sigma \sim 1$.

In the most general nonrelativistic case of a thermal quantum plasma, of which the classical thermal plasma is a special case, the secondary conductivity quantity σ depends on the dimensional primary quantities $a_1 = e$(electron charge), $a_2 = m$(electron mass), $a_3 = \hbar$(Planck's constant), $a_4 = n$(electron density), and $a_5 = KT$(thermal energy). The dimensionless constant C_σ is in general a function of the dimensionless parameters p/p_i of the plasma,

$$C_\sigma = C_\sigma(Z_i, m/m_i, \dots p/p_i, \dots) \quad . \tag{4}$$

E.g., Z_i is the ratio of the magnitudes of the ion and electron charges, m/m_i is the electron to ion mass ratio, etc. The electrical conductivity σ and its primary quantities have the following dimensions $\mathcal{D}$(L = dimension of length, T = dimension of time, M = dimension of mass) and units $\mathcal{U}$ in the cgs-system:

$$\begin{aligned}
\mathcal{D}[\sigma] &= T^{-1} , & \mathcal{U}[\sigma] &= \mathrm{sec}^{-1} ;\\
\mathcal{D}[e] &= L^{3/2} M^{1/2} T^{-1} , & \mathcal{U}[e] &= \mathrm{cm}^{3/2}\,\mathrm{gr}^{1/2}\,\mathrm{sec}^{-1} ;\\
\mathcal{D}[m] &= M , & \mathcal{U}[m] &= \mathrm{gr} ;\\
\mathcal{D}[\hbar] &= ML^2T^{-1} , & \mathcal{U}[\hbar] &= \mathrm{gr\ cm^2\ sec^{-1}} ;\\
\mathcal{D}[n] &= L^{-3} , & \mathcal{U}[n] &= \mathrm{cm}^{-3} ;\\
\mathcal{D}[KT] &= ML^2T^{-2} , & \mathcal{U}[KT] &= \mathrm{gr\ cm^2\ sec^{-2}} .
\end{aligned} \tag{5}$$

It is recognized that the secondary and primary quantities depend only on three basic dimensions which are independent, namely L, T, and M. For this reason, dimensional analysis provides at most three independent equations for the determination of the powers N_i, i = 1,2,3,...M in Eq. (3). That is, at most three powers can be calculated while at least M-3 powers have to be determined by comparison with experiments or by physical arguments.

Two of the primary quantities are variables, namely n and KT ($K = 1.380 \times 10^{-16}$ gr cm^{2}sec^{-2}/^{o}K), whereas the remaining three primary quantities are elementary constants ($e = 4.803 \times 10^{-10}$ cm$^{3/2}$ gr$^{1/2}$ sec^{-1}, $m = 9.109 \times 10^{-28}$ gr, $\hbar = 1.054 \times 10^{-27}$ gr cm^{2}sec^{-1}).

The fundamental equation (3) is applied below to the determination of the electrical conductivity of (nonrelativistic) nonideal, classical and quantum plasmas, in which the electrons are responsible for the electric current transport. To illustrate the results, they will be expressed in terms of the interaction parameter γ for Z = 1 and the order-of-magnitude of fundamental electron energies:

$$\gamma_T = E_C/E_T = ne^2/KT \quad , \tag{6}$$

$$E_T = KT \quad , \tag{7}$$

$$E_C = e^2 n^{1/3} \quad , \tag{8}$$

$$E_Q = \hbar^2/mn^{-2/3} \quad . \tag{9}$$

E_Q is the order-of-magnitude of the quantum potential energy $Q = -(\hbar^2/2m)\nabla^2\rho^{1/2}/\rho^{1/2}$ of the electron in the plasma. E_C and E_T are the Coulomb interaction and thermal energies.

CLASSICAL T=0 PLASMA

In a classical ($\hbar \to 0$) electron plasma, in which the thermal energy KT is negligible compared with the Coulomb interaction energy, the conductivity depends on the dimensional parameters e, m, and n. By Eq. (3),

$$\sigma = C_\sigma \, e^{N_1} m^{N_2} n^{N_3} \quad . \tag{10}$$

Hence,

$$T^{-1} = (L^{3/2} M^{1/2} T^{-1})^{N_1} M^{N_2} L^{-3N_3} \quad , \tag{11}$$

i.e.,

$$(3/2)N_1 - 3N_3 = 0, \quad (1/2)N_1 + N_2 = 0, \quad -N_1 = -1. \tag{12}$$

These are three independent equations, which determine the powers N_1, N_2, and N_3 uniquely,

$$N_1 = 1, \quad N_2 = -1/2, \quad N_3 = 1/2 \ . \tag{13}$$

By Eqs. (10) and (13), the conductivity of the zero-temperature, classical plasma is

$$\sigma = C_\sigma \, e n^{1/2}/m^{1/2} \quad . \tag{14}$$

Eq. (14) indicates that $\sigma = C_\sigma \omega_p/(4\pi)^{1/2}$ increases proportional to $n^{1/2}$, where $\omega_p = (4\pi n e^2/m)^{1/2}$ is the plasma frequency. This result was first derived by Buneman[15] and later by Hamberger and Friedman[16] for an electrostatically turbulent T = 0 plasma by means of semi-quantitative physical arguments, which give $C_\sigma = \pi^{1/2}(m_i/m)^{1/3}$ for Z = 1.

CLASSICAL T>0 PLASMA

In a classical ($h \to 0$), thermal ($KT>0$) electron plasma, the conductuvity depends on the dimensional parameters e, m, n, and KT. By Eq. (3)

$$\sigma = C_\sigma \, e^{N_1} m^{N_2} n^{N_3} (KT)^{N_4} \quad . \tag{15}$$

Hence,

$$T^{-1} = (L^{3/2} M^{1/2} T^{-1})^{N_1} M^{N_2} L^{-3N_3} (ML^2 T^{-2})^{N_4} \quad , \tag{16}$$

i.e.,

$$\frac{3}{2}N_1 - 3N_3 + 2N_4 = 0, \quad \frac{1}{2}N_1 + N_2 + N_4 = 0, \quad -N_1 - 2N_4 = -1 \quad . \tag{17}$$

These are three independent equations, which determine three of the four powers in terms of the fourth,

$$N_1 = 1 - 2N, \quad N_2 = -\frac{1}{2} \; , \quad N_3 = \frac{1}{2} - \frac{1}{3}N, \quad N_4 = N \; . \tag{18}$$

According to Eqs. (15) and (18), the conductivity of the classical, thermal plasma is

$$\sigma = C_\sigma \, e^{1-2N} m^{-1/2} \, n^{\frac{1}{2} - \frac{1}{3}N} \, (KT)^N \quad . \tag{19}$$

Collecting of powers of N reveals the dimensionless group contained in Eq. (19), and condenses the conductivity formula to

$$\sigma = C_\sigma (KT/e^2 n^{1/3})^N \, e n^{1/2}/m^{1/2} \quad . \tag{20}$$

For an ideal (classical) plasma, $\sigma = (ne^2/m)\tau$ cannot depend on n since $\tau \propto n^{-1}$ for binary e-i collisions. Hence, σ and N are for an ideal $T>0$ plasma

$$\sigma = C_\sigma \, (KT)^{3/2}/e^2 m^{1/2} \quad , \qquad N = 3/2 \; , \tag{21}$$

in agreement with kinetic theory[12], which shows that $C_\sigma = 3/4(2\pi)^{1/2} Z \ell n \Lambda \sim 10^{-1}$. For N = 0, Eq. (20) reduces to σ of the classical $T = 0$ plasma, Eq. (14).

For a nonideal (classical) plasma with ν-body interactions, we have $\tau \propto n^{1-\nu}$ and $\sigma = (ne^2/m)\tau \propto n^{2-\nu}$, with $\nu > 2$, i.e. $N = 3\nu - 9/2 > 3/2$. Thus, we find for the conductivity of nonideal, classical plasmas

$$\sigma = C_\sigma \gamma_T^{-N} e n^{1/2}/m^{1/2} \quad , \quad N > 3/2 \ , \tag{22}$$

where $\gamma_T = E_C/E_T$ is the nonideality parameter defined in Eqs. (6) - (8). Eq. (22) expresses the important result that the conductivity of a nonideal, classical plasma, $\sigma = C_\sigma \gamma_T^{-N} \omega_p/2\pi^{1/2}$, decreases proportional to γ_T^{-N} with increasing nonideality γ_T since $N > 3/2$. The exact value of N can be determined by comparison with experimental data.

T = 0 QUANTUM PLASMA

In a completely degenerate electron plasma, $E_T << E_Q$, the conductivity depends on the dimensional parameters e, m, n, and $\hbar$, but not on KT. By Eq. (3),

$$\sigma = C_\sigma \, e^{N_1} m^{N_2} n^{N_3} \hbar^{N_4} \quad . \tag{23}$$

Hence,

$$T^{-1} = (L^{3/2} M^{1/2} T^{-1})^{N_1} M^{N_2} L^{-3N_3} (ML^2 T^{-1})^{N_4} \quad , \tag{24}$$

i.e.,

$$\tfrac{3}{2}N_1 - 3N_3 + 2N_4 = 0, \quad \tfrac{1}{2}N_1 + N_2 + N_4 = 0, \quad -N_1 - N_4 = -1. \tag{25}$$

These are three independent equations, which determine three of the four powers in terms of the fourth,

$$N_1 = 1-2N, \quad N_2 = -\frac{1}{2} - N, \quad N_3 = \frac{1}{2} + \frac{1}{3}N, \quad N_4 \equiv 2N. \tag{26}$$

Substitution of Eq. (26) into (23) yields as conductivity of the completely degenerate electron plasma

$$\sigma = C_\sigma \left(\frac{\hbar^2 n^{1/3}}{m e^2}\right)^N e\, n^{1/2}/m^{1/2} \quad . \tag{27}$$

For N = 3/2, Eq. (27) leads to the conductivity of a solid metal at T = 0,

$$\sigma = C_\sigma \, \hbar^3 n/e^2 m^2 \quad , \qquad N = 3/2 \quad , \tag{28}$$

where[17] $C_\sigma \propto Z^{-1}$. For N=0, Eq. (27) reduces to σ of the classical T=0 plasma, Eq. (14).

For complete degeneracy, E_T is negligible compared with $E_Q \sim \frac{1}{2} m v_F^2$, where $v_F \sim (\hbar/m) n^{1/3}$ is the Fermi velocity. For this reason, Eq. (27) is rewritten in the form

$$\sigma = C_\sigma \, \gamma_Q^{-N} \, e n^{1/2}/m^{1/2} \tag{29}$$

where

$$\gamma_Q = E_C/E_Q = e^2 n^{1/3}/(\hbar^2/mn^{-2/3}) \tag{30}$$

is the nonideality parameter of the completely degenerate electron plasma for $Z = 1$. Since $N = 3/2$ for the $T=0$ metal, it is to be expected that $N>0$ for the completely degenerate electron plasma, i.e. its conductivity decreases with increasing nonideality γ_Q, Eq. (29). This formula is useful for the interpretation of conductivity data of completely degenerate electron plasmas, with N as adjustable parameter.

T>0 QUANTUM PLASMA

In a partially degenerate electron plasma, $E_T \lessgtr E_Q$, the conductivity depends on the dimensional parameters e, m, n, KT, and $\hbar$. By Eq. (3),

$$\sigma = C_\sigma \, e^{N_1} m^{N_2} n^{N_3} (KT)^{N_4} \hbar^{N_5} \quad . \tag{31}$$

Hence,

$$T^{-1} = (L^{3/2} M^{1/2} T^{-1})^{N_1} M^{N_2} L^{-3N_3} (ML^2T^{-2})^{N_4} (ML^2T^{-1})^{N_5} \quad , \tag{32}$$

i.e.,

$$\begin{aligned} \tfrac{3}{2}N_1 - 3N_3 + 2N_4 + 2N_5 &= 0 \quad , \\ \tfrac{1}{2}N_1 + N_2 + N_4 + N_5 &= 0 \quad , \\ -N_1 - 2N_4 - N_5 &= -1 \quad . \end{aligned} \tag{33}$$

These are three independent equations, which permit to express three of the five powers in terms of the remaining powers,

$$\begin{aligned} &N_1 = 1 - 2A - 2B \; , \quad N_2 = -\tfrac{1}{2} - B, \quad N_3 = \tfrac{1}{2} - \tfrac{1}{3}A + \tfrac{1}{3}B, \\ &N_4 \equiv A, \quad N_5 \equiv 2B \quad . \end{aligned} \tag{34}$$

Combining of Eq. (31) with (34) gives for the conductivity of the T>0 quantum plasma

$$\sigma = C_\sigma \left(\frac{KT}{e^2 n^{1/3}}\right)^A \left(\frac{\hbar^2 n^{1/3}}{me^2}\right)^B en^{1/2}/m^{1/2} \tag{35}$$

where A and B are powers which can not be determined by dimensional reasoning.

For A = -1 and B = -3/2, Eq. (35) yields the conductivity of solid metals at temperatures T>0,

$$\sigma = C_\sigma \frac{me^6}{\hbar^3} \frac{n^{1/3}}{KT} \quad , \qquad A = -1, \qquad B = -3/2, \tag{36}$$

where[17] $C_\sigma \propto Z^{-1/3}$. Eq. (36) expresses the 1/T - law[17] of the metallic conductivity at "high temperatures".

Eq. (35) contains the dimensionless groups γ_T and γ_Q, which permit to rewrite the conductivity formula as

$$\sigma = C_\sigma \, \gamma_T^{-A} \, \gamma_Q^{-B} \, en^{1/2}/m^{1/2}. \tag{37}$$

Since $A = -1$ for $T>0$ metals [Eq. (36)] and $B = 3/2$ for $T=0$ metals[Eq. (28)], one can speculate that $A \lessgtr -1$ and $B>0$ for nonideal, partially degenerate plasmas. A and B can readily be determined by means of conductivity measurements for nonideal quantum plasmas. The theoretical determination of the nonlinear dependence of σ on γ_T and γ_Q in Eq. (37) from a physical model of more than two-body interactions is left to future research.

In conclusion, it is noted that we have derived new conductivity formulas for nonideal classical $T>0$ plasmas [Eq. (20) or (22)], completely degenerate plasmas [Eq. (27) or (29)], and nonideal $T>0$ quantum plasmas [Eq. (35) or (37)]. These formulas can be used to interprete conductivity measurements on nonideal plasmas. Once the still undetermined powers N, A, and B are known empirically, it should be possible to develop a conductivity theory for nonideal plasmas which provides an explanation of the γ_T and γ_Q dependence from first principles.

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III. ELECTRICAL CONDUCTIVITY OF NONIDEAL PLASMA

ABSTRACT

The electrical conductivity of fully ionized, moderately nonideal plasmas with interaction parameters $0.1 < \gamma \leq 1$, where $\gamma = Ze^2 n^{1/3}/KT$ is the ratio of Coulomb and thermal energies, is calculated for displaced Maxwell and Fermi electron distributions, respectively. The electrons are scattered by an effective Coulomb potential $\phi(r) = Zer^{-1} \exp(-r/\delta)$, with $\delta = (3n/4\pi Z)^{-1/3}$ the mean ion distance, which considers binary $(0 < r \leq \delta)$ and many-body $(\delta < r < \infty)$ interactions. It is shown that the resulting conductivity formula is applicable to densities up to four orders of magnitude higher than those of the ideal conductivity theory, which breaks down at higher densities because the Debye radius loses its physical meaning as a shielding length and upper impact parameter. In the limit of complete degeneracy, the conductivity formula reduces to that of a solid metal.

INTRODUCTION

The theory of the electrical conductivity of fully ionized plasmas[1-3] based on the Boltzmann equation, the Fokker-Planck equation (derived by expanding the binary collision integral for the small, successive velocity changes of Coulomb scattering), or the Lenard-Balescu equation (taking into account the dielectric properties of the medium) is in agreement with the experimental data for rarefied high-temperature plasmas, $\gamma \ll 1$. The interaction parameter is defined as the ratio of (average) Coulomb interaction ($Ze^2n^{1/3}$) and thermal (KT) energies (n is the electron density and Z the ion charge number),

$$\gamma = Ze^2n^{1/3}/KT = 1.670 \times 10^{-3}Zn^{1/3}/T$$

in cgs-units which will be used throughout. The conventional transport calculations[1-3] give an electrical conductivity of the form $\sigma \sim (KT)^{3/2}/m^{1/2}e^2Z\ell n\Lambda$ for classical ideal plasmas, where $\Lambda = [1 + (D/p_o)^2]^{1/2} \cong D/p_o$ for $D \gg p_o$. D is the maximum impact parameter (Debye length), $D = [KT/4\pi e^2(1 + Z)n]^{1/2}$, and p_o is the average impact parameter for 90° deflections (Landau length), $p_o = Ze^2/2KT$. The condition, $\Lambda \gg 1$ or $\ell n\Lambda \sim 10^1$ is satisfied only for not too low temperatures T and not too high densities n.[4] Conductivity formulas with this Coulomb logarithm break/down for large interaction parameters γ and densities n, since the Debye radius

$$D = [Z/4\pi(1 + Z)]^{1/2}\gamma^{-1/2}n^{-1/3}$$

becomes smaller than the atomic dimension 10^{-8}cm and, thus, completely loses its physical meaning as an electric shielding length and maximum impact parameter. E.g., for $T = 10^4$ °K, $\gamma > 10^0$ and $D < 10^{-8}$cm if $n > 10^{20}$cm^{-3}. Moderately nonideal plasmas with $\gamma \sim 1$ are readily generated through shock wave compression and exhibit conductivities of the order $\sigma \sim 10^1 - 10^2$mho/cm[5-6], which are much smaller than those which would be obtained by applying the conductivity formula for ideal plasmas in the nonideal regime.

Although there are some bulk measurements of the electrical conductivity of nonideal cesium and noble gas plasmas available[5-8], theoretical explanations of these results are still missing. The momentum and energy transport in weakly nonideal plasmas, $\gamma \ll 1$, was treated by Wilhelm[9] by means of an experimentally shielded Coulomb potential, which permits to consider not only short-range binary ($r \leq D$) but also long-range many-body ($r > D$) interactions. This interaction model was used shortly afterwards by Rogov[10] for the calculation of the conductivity of weakly, nonideal argon and xenon plasmas with Debye shielding.

For moderately nonideal plasmas, $0.1 < \gamma \leq 1$, various phenomenological approaches have been used to extend the conductivity formula of ideal plasmas, e.g., Goldbach et al [11] multiply the Debye length D with a free parameter x(p) which is chosen to match the experimental data, i.e. to compensate for the too rapid decrease of D with pressure. A kinetic equation has been proposed for nonideal plasmas by Klimontovich[12], which considers spatial correlations and temporal retardation in the collision integrals. This equation has not yet lead to transport coefficients because of the mathematical difficulties associated with its solution.

In the following, the momentum relaxation time and the electrical conductivity of (i) classical and (ii) degenerate plasmas is calculated for intermediate non-ideal conditions, $0.1 < \gamma \leq 1$. For this region of interaction, the concept of Debye shielding already breaks down since the number of particles in the Debye sphere $4\pi D^3/3$ is no longer large compared with one for $\gamma > 0.1$. This difficulty can not be remidied by replacing D with the quantum mechanical shielding length (even when thermal effects are included)[13]

$$D_F = (\pi a_o/k_F)^{1/2}, \quad a_o = \frac{\hbar^2}{me^2}, \quad k_F = (\frac{3n}{8\pi})^{1/3}$$

which is of the same order as D in most high pressure plasmas, e.g. $D_F \sim 10^{-8}$ cm for $n = 10^{20}$ cm^{-3}. From the definition of the mean particle distance, it is clear that the mean ion distance $\delta \sim n_i^{-1/3}$ separates the region in which an electron experiences few - body encounters ($r \lesssim \delta$) from the region in which an electron experiences many - body interactions ($r > \delta$) in a nonideal plasmas, as long as $\delta > 10^{-8}$ cm ($n_i < 10^{24}$ cm^{-3}). Thus, the mean ion distance evolves naturally as the characteristic interaction distance for nonideal plasmas, for which Debye and Fermi shielding fail.

We calculate first the electrical conductivity of plasmas with (i) Maxwell and (ii) Fermi distributions of the electrons, when all ions have the same charge number. Then, we generalize the conductivity formulas for plasmas with several ion components of charge number Z_i, $i = 1, 2, \ldots$. The electrons are assumed to be scattered by the exponentially decaying Coulomb potential $\phi = Zer^{-1} \exp(-r/\delta)$ which takes many - body interactions at distances $r > \delta$ into account. The considerations are applicable only to moderately nonideal conditions, $0.1 < \gamma \lesssim 1$, up to densities $n \ll 10^{24}$ cm.

For solid state densities and larger densities, $n \gtrsim 10^{24}$ cm^{-3}, we have $\delta < 10^{-8}$ cm and the chosen interaction potential $\phi(r)$ is no longer valid. For densities $n \gtrsim 10^{24}$ cm^{-3}, the Coulomb field of the ions is changed by polarization of the bound electron cloud, so that the free electrons are scattered as in solids by the ions. For this reason, plasmas in the solid phase are not treated herein.

PHYSICAL FOUNDATIONS

The electrical conductivity σ of any gaseous, liquid, or solid medium, in which the electrical current transport is due to electrons, is proportional to the electron density n and the relaxation time τ of the average momentum $\langle m\vec{v}_e \rangle$ of the electrons (m is the electron mass and e > 0 is the elementary charge)

$$\sigma = (ne^2/m)\,\tau \quad . \tag{1}$$

Eq.(1) reduces the calculation of σ to the evaluation of τ. The relaxation time τ is determined by the scattering potential and the (classical or quantum statistical) kinetics of the electron gas in the electric field.

In proper nonideal plasmas, the region $0 < r \lesssim \delta$ of binary and few - body collisons and the region $\delta < r < \infty$ of many - body interactions are bounded by the mean ion radius,

$$(4\pi\delta^3/3)n_i = 1, \qquad \delta = (3Z/4\pi n)^{1/3}, \tag{2}$$

since electric Debye shielding exists only for weakly nonideal conditions, $\gamma \ll 1$. For this reason, the effective Coulomb potential of Z times charged ions is in plasmas of intermediate nonideality

$$\phi(r) = Zer^{-1}\exp(-r/\delta), \quad 0 < r < \infty \quad , \qquad 0.1 < \gamma \lesssim 1. \tag{3}$$

Eq.(3) is no longer applicable to plasmas with densities comparable to solids. It contains the binary and few - body collisions at distances $0 < r \lesssim \delta$ and the many - body interactions at distances $\delta < r < \infty$.

The differential cross section $\sigma(\theta,g)$ for the scattering $(\vec{g} \to \vec{g}\,^*)$ of electrons by the potential (3) is in the center of mass system[14)]

$$\sigma(\theta,g) = (Ze^2/2m)^2/[g^2 \sin^2(\theta/2) + u^2]^2 \tag{4}$$

where

$$\theta = \sphericalangle(\vec{g},\vec{g}^*), \quad \vec{g} = \vec{v}_e - \vec{v}_i, \quad g^* = \vec{v}_e^* - \vec{v}_i^* \tag{5}$$

and

$$u = \hbar/2m\delta \quad . \tag{6}$$

The electron and ion velocities before and after the interaction are designated by $\vec{v}_{e,i}$ and $\vec{v}^{*}_{e,i}$, respectively. The speed u corresponds to a de Broglie wave length of the order $\lambda \sim \delta$. For $u \to 0$ or $u \to \infty$, Eq.(4) reduces to the Rutherford cross section.

The scattering cross section $\sigma(\theta,g)$ is strictly valid only in the Born approximation[15)]. Contrary to what one might expect in general for the latter, Eq.(4) describes in good approximation the scattering in the exponentially decaying potential (3) because of the peculiarity of the Coulomb interaction. The Coulomb interation $\phi \sim 1/r$ has the unique property that the Born approximation and the exact wave mechanical approach give the same scattering cross section[15] (identical with the Rutherford formula). In the region $0 < r < \delta$, the interaction potential (1) is practically Coulombic, and thus the Born approximation gives the correct solution. In the region $\delta < r < \infty$, the interaction potential (1) is effectively screened, i.e., the Born approximation gives the correct solution because $\phi(r)$ is small. In the transition zone $r \sim \delta$, the Born approximation holds fairly well for reasons of continuity.

The relaxation time τ is obtained by evaluation of the collision integrals for the electron momentum $m\vec{v}_e$ for the (i) classical and (ii) degenerate plasma, respectively. Both in the cases of classical and Fermi statistics, the particle velocities $\vec{v}_{e,i}$ and $\vec{v}^{*}_{e,i}$ before and after the interaction are interrelated by the conservation equations for momentum and energy.

CONDUCTIVITY OF CLASSICAL PLASMA

According to kinetic theory, the average momentum density $n\,m(\langle\vec{v}_e\rangle - \langle\vec{v}_i\rangle)$ exchanged per unit time between electrons and ions, interacting with the Coulomb potential (1), is given by the collision integral for $m\vec{v}_e$, which determines the momentum relaxation time τ,

$$-nm(\langle\vec{v}_e\rangle - \langle\vec{v}_i\rangle)/\tau =$$
$$m\int\ldots\int \vec{v}_e\{f_e(\vec{v}_e^*)\,f_i(\vec{v}_i^*) - f_e(\vec{v}_e)\,f_i(\vec{v}_i)\}g\,\sigma(\theta,g)\,d\Omega\,d\vec{v}_e\,d\vec{v}_i \quad . \tag{7}$$

The scattering cross section $\sigma(\theta,g)$ is given in Eq.(4) and the solid angle element is $d\Omega = \sin\theta\,d\theta\,d\phi$. In response to an applied electric field $\vec{E}$, the electrons and ions drift with velocities $\langle\vec{v}_e\rangle$ and $\langle\vec{v}_i\rangle$ so that their distribution functions are displaced Maxwellians,

$$f_s(\vec{v}_s) = n_s(m_s/2\pi KT_s)^{3/2}\exp[-\tfrac{1}{2}m_s(\vec{v}_s - \langle\vec{v}_s\rangle)^2/KT_s], \quad s=e,i. \tag{8}$$

Eq.(8) represents a 5-moment-approximation to the nonequilibrium solution of the Boltzmann equation.[16] The perturbations of $f_s(\vec{v}_s)$ due to viscous stresses and heat flows are neglected in Eq.(8), since they yield only corrections of higher order to the conductivity.

The collision integral (7) is integrated by standard methods[9] for subsonic drift velocities, $|\langle\vec{v}_e\rangle - \langle\vec{v}_i\rangle| < (2KT/m)^{1/2}$, with the usual approximations ($m_{ei} = m_e m_i/(m_e + m_i) \simeq m_e \equiv m$, $T_{es} = m_{es}[(T_e/m_e) + (T_i/m_i)] \simeq T_e \equiv T$). For supersonic drift velocities, a linear response $\vec{j} = \sigma\vec{E}$ between current density $\vec{j}$ and electric field $\vec{E}$ does no longer exist.[9] The resulting relaxation time is given by:[17]

$$\tau^{-1} = \frac{8}{3}(2KT/\pi m)^{1/2}\,n_i\,Q \quad , \tag{9}$$

$$Q = \frac{\pi}{4}(Ze^2/KT)^2\,L \quad , \tag{10}$$

$$L = e^{\Lambda^{-1}}E_1(\Lambda^{-1}) \quad , \tag{11}$$

where

$$\Lambda = 2KT/mu^2 = (8m/h^2)(4\pi n/3Z)^{-2/3}KT \tag{12}$$

and

$$E_1(x) = -\Gamma - \ell n\, x - \sum_{m=1}^{\infty} (-1)^m x^m/m(m!) \tag{13}$$

is the exponential integral of order one (Γ = 0.477 ... = Euler's constant).[18] The latter satisfies the inequalities, for $x > 0$,[18]

$$\frac{1}{2}\ell n(1 + 2/x) < e^x E_1(x) < \ell n(1 + 1/x), \qquad (1 + x)^{-1} < e^x E_1(x) < x^{-1}. \tag{14}$$

Accordingly, Eq.(11) gives formally for small and large arguments $x = \Lambda^{-1}$,

$$L \simeq \ell n\, \Lambda \quad , \qquad \Lambda >> 1 \quad , \tag{15}$$

$$L \simeq \Lambda \quad , \qquad \Lambda << 1 \quad . \tag{16}$$

Combining of Eqs. (9) - (12) with Eq. (1) yields the desired electric conductivity of the classical plasma of intermediate nonideality, $0.1 < \gamma \leq 1$:

$$\sigma = 3(KT)^{3/2}/2(2\pi m)^{1/2}\, e^2\, ZL \tag{17}$$

where

$$L = \ell n[8m\hbar^{-2}(4\pi n/3Z)^{-2/3}\, KT], \qquad \Lambda >> 1, \tag{18}$$

$$L = 8m\hbar^{-2}(4\pi n/3Z)^{-2/3}\, KT \;, \qquad \Lambda << 1. \tag{19}$$

by Eqs. (15) - (16).It is instructive to reqrite Λ in terms of the thermal and quantum potential energies of an electron,

$$\Lambda = 8E_T/E_Q, \qquad E_T = KT, \qquad E_Q = \hbar^2/m\delta^2 \quad . \tag{20}$$

The conductivity formula (17) differs from the conductivity of the ideal plasma[1-3] mainly through the term L. The latter has the form of a Coulomb logarithm, $L = \ell n\, \Lambda$, for $\Lambda >> 1$, i.e. for all densities n and temperatures T for which the plasma is nondegenerate, $E_T > E_Q$, Eq.(20). Numerically,

$$\Lambda = 3.482 \times 10^{11}(n/Z)^{-2/3}\, T \quad . \tag{21}$$

The corresponding argument $\Lambda_D = 2KT\, D/Ze^2$ of the ideal Coulomb logarithm[1-3], $\ell n\, \Lambda_D$, is

$$\Lambda_D = 1.464 \times 10^4\, Z^{-1}(1 + Z)^{1/2}\, n^{1/2}\, T^{3/2} \quad . \tag{22}$$

Table I compares Λ of the nonideal plasma and Λ_D of the ideal plasma for large densities n and the typical temperature $T = 10^4$ °K. It is seen that $\ln \Lambda_D$ of the ideal plasma is unacceptably small for densities $n \gtrsim 10^{18}$ cm^{-3}, whereas $\ln \Lambda$ of the nonideal plasma has reasonable values up to densities $n \lesssim 10^{22}$ cm^{-3}, if $T = 10^4$ °K. The conductivity formula (17) holds, therefore, for densities n up to 4 orders of magnitude higher than the conductivity formula of the ideal plasma. Eq.(17) is not applicable to n - T regions for which $\Lambda << 1$, i.e. $E_T << E_Q$, Eq.(20), since in this case the electrons would be degenerate.

<u>TABLE I:</u> Λ and Λ_D versus n for $T = 10^4$ °K and Z = 1.

n[cm^{-3}]	10^{18}	10^{20}	10^{22}	10^{24}
Λ	3.482×10^3	1.616×10^2	0.750×10^2	0.348×10^0
Λ_D	1.035×10^1	1.035×10^0	1.035×10^{-1}	1.035×10^{-2}

The conductivity formula (17) becomes in cgs - units or practical units (9×10^{11} sec^{-1} = 1 mho cm^{-1}),

$$\sigma = 1.394 \times 10^8 \, T^{3/2} / Z \ln \Lambda \, [\mathrm{sec}^{-1}] = 1.549 \times 10^{-4} \, T^{3/2} / Z \ln \Lambda \, [\mathrm{mho\ cm}^{-1}] \tag{23}$$

where Λ is given in Eq.(21). Accordingly, if $T = 10^4$ °K and $Z = 1$, $\sigma = 1.899 \times 10^1$ mho cm^{-1} for $n = 10^{18}$ cm^{-3} and $\sigma = 2.097 \times 10^1$ mho cm^{-1} for $n = 10^{20}$ cm^{-3}.

CONDUCTIVITY OF QUANTUM PLASMA

The electrons in a plasma become degenerate if their thermal DeBroglie wave length is larger than the mean electron distance, i.e. at densities ($h=2\pi\hbar$)

$$n > 2(2\pi m\, KT/h^2)^{3/2} = 4.828 \times 10^{15}\, T^{3/2}. \tag{24}$$

E.g., for $T = 10^4$ °K, degeneracy requires $n > 5 \times 10^{21}$ cm^{-3}. In view of their large mass $m_i >> m$, the ions can be treated as classical. The momentum relaxation time τ of the degenerate electron gas is determined by the quantum statistical collision integral for $m\vec{v}_e$,[19]

$$- nm(\langle\vec{v}_e\rangle - \langle\vec{v}_i\rangle)/\tau =$$

$$m \int \ldots \int \vec{v}_e \{ f_e(\vec{v}_e^*) f_i(\vec{v}_i^*) [1 - \tfrac{1}{2}h^3\, f_e(\vec{v}_e)] - f_e(\vec{v}_e)\, f_i(\vec{v}_i) [1 - \tfrac{1}{2}h^3 f_e(\vec{v}_e^*)]$$

$$\times \quad g\, \sigma(\theta, g)\, d\Omega \quad d\vec{v}_e\, d\vec{v}_i \tag{25}$$

where the scattering cross section $\sigma(\theta,g)$ between electrons and ions is given by Eq.(4). The solutions to the velocity distributions are the displaced Maxwellian (8) for the ions (s = i) and the 5-moment Fermi approximation for the electrons,

$$f_e(\vec{v}_e) = 2(m/h)^3\ \{1 + \exp[\tfrac{1}{2}m(\vec{v}_e - \langle\vec{v}_e\rangle)^2 - \mu]/KT\}^{-1}\ . \tag{26}$$

The chemical potential $\mu = \mu(n, T)$ is determined by the integral functional $n = \int f_e(\vec{c}_e;\ \mu)\ d\vec{c}_e$.

Again, a linear response $\vec{j} = \sigma\vec{E}$ exists for small drift velocities $\langle\vec{v}_{e,i}\rangle$ or weak electric fields $\vec{E}$. Integration of Eq.(25) yields, after standard approximations, for the relaxation time of the degenerate electron gas:

$$\tau^{-1} = \frac{8}{3}\left(\frac{2}{m}\right)^{1/2} \frac{e^4 Z^2 n_i\, L_Q}{(KT)^{3/2} R(n,T)} \tag{27}$$

where

$$L_Q = e^{\Lambda_Q^{-1}}\, E_1(\Lambda_Q^{-1}), \tag{28}$$

$$\Lambda_Q = \frac{16}{3}\ \frac{m}{\hbar^2}\ \left(\frac{4\pi n}{3Z}\right)^{-2/3} KT\ Q(n,T) \tag{29}$$

and

$$Q(n,T) = \frac{3}{2}(1 + 2^{-2}\frac{n}{\tilde{n}} + \ldots), \qquad n << \tilde{n}(T), \tag{30}$$

$$R(n,T) = \frac{4}{\pi^{1/2}}(1 + 2^{-2}\frac{n}{\tilde{n}} + \ldots), \qquad n << \tilde{n}(T), \tag{31}$$

but

$$Q(n,T) = \frac{3}{5}(\frac{3\pi^{1/2}}{4}\frac{n}{\tilde{n}})^{2/3}[1 + \frac{5\pi^2}{12}(\frac{3\pi^{1/2}}{4}\frac{n}{\tilde{n}})^{-4/3} + \ldots], \qquad n >> \tilde{n}(T), \tag{32}$$

$$R(n,T) = \frac{1}{2}(\frac{3\pi^{1/2}}{4}\frac{n}{\tilde{n}})^{1}[1 + \frac{3\pi^2}{4}(\frac{3\pi^{1/2}}{4}\frac{n}{\tilde{n}})^{-4/3} + \ldots], \qquad n >> \tilde{n}(T). \tag{33}$$

Equations (30) - (31) and Eqs. (32) - (33) result from expansions of the Fermi distribution (26) in the collision integral (25) for densities $n << \tilde{n}(T)$ and $n >> \tilde{n}(T)$, where

$$\tilde{n}(T) = 2(2\pi m\, KT/h^2)^{3/2} \tag{34}$$

is the critical density which separates the degenerate and non-degenerate regimes. These series are based on expansions of the normalization integral of Eq.(26), which gives the chemical potential μ explicitly as a function of n and T,

$$\frac{\mu}{KT} = \ell n\{\frac{n}{\tilde{n}}[1 + 2^{-3/2}(\frac{n}{\tilde{n}})^{1} + (\frac{1}{4} - 3^{-3/2})(\frac{n}{\tilde{n}})^{2} + \ldots], \qquad n < \tilde{n}(T), \tag{35}$$

$$\frac{\mu}{KT} = (\frac{3\pi^{1/2}}{4}\frac{n}{\tilde{n}})^{2/3}[1 - \frac{\pi^2}{12}(\frac{3\pi^{1/2}}{4}\frac{n}{\tilde{n}})^{-4/3} - \frac{\pi^4}{80}(\frac{3\pi^{1/2}}{4}\frac{n}{\tilde{n}})^{-8/3} + \ldots], \qquad n > \tilde{n}(T) \tag{36}$$

Combining of the conductivity formula in Eq.(1) with the relaxation time of Eq(27) yields for the electrical conductivity of the degenerate electron plasma of intermediate nonideality, $0.1 < \Upsilon \lesssim 1$:

$$\sigma = \frac{3(KT)^{3/2}R(n,T)}{8(2m)^{1/2}e^2 Z\, L_Q} \tag{37}$$

where L_Q is given by Eq.(29). In the limiting cases of large and small values of Λ_Q,

$$L_Q = \ell n[\frac{2}{3}\Lambda\, Q(n,T)], \qquad \Lambda Q >> 1, \tag{38}$$

$$L_Q = \frac{2}{3}\Lambda Q(n,T), \qquad \Lambda Q << 1, \tag{39}$$

since $\Lambda_Q = (2/3)\Lambda Q$ by comparison of Eqs.(29) and (12).

For $n/\tilde{n} \to 0$, Eqs.(37) and (38) reduce to the classical conductivity, Eqs. (17) and (18), since $R(n,T) \to 4/\sqrt{\pi}$ and $Q(n,T) \to 3/2$ for $n/n \to 0$ by Eqs.(30) – (31). On the other hand, Eqs. (37) and (39) give in the limit of complete degeneracy, $n/\tilde{n} \to \infty$:

$$\sigma = \frac{9h^3 n}{2^9 \pi m^2 e^2 Z^2 L_Q} \quad , \tag{40}$$

where

$$L_Q = \frac{3}{40}\left(\frac{3}{2\pi}\right)^{1/3} \frac{Z^{2/3}}{\pi^3} \quad , \quad L_Q \ll 1 \quad , \tag{41}$$

by Eqs. (32) and (33). Euqations (40) ~ (41) combine to the conductivity formula

$$\sigma = \frac{15\pi^2}{2^6} \left(\frac{2\pi}{3}\right)^{1/3} \frac{h^3 n}{m^2 e^2 Z^{5/3}} \quad . \tag{42}$$

Equation (42) agrees with the expression for the conductivity of a low temperature metal.[20]

For numerical evaluations, Eq.(42) is stated in cgs units (sec^{-1}) and practical units ($mho\ cm^{-1}$),

$$\sigma = 4.498 \times 10^{-6}\ Z^{-5/3}\ n\ [sec^{-1}] = 4.998 \times 10^{-18}\ Z^{-5/3}\ n\ [mho\ cm^{-1}]. \tag{43}$$

Accordingly, $\sigma \simeq 4 \times 10^4\ mho\ cm^{-1}$ for $n = 10^{22}\ cm^{-3}$ and $\sigma \simeq 5 \times 10^6\ mho\ cm^{-1}$ for $n = 10^{24}\ cm^{-3}$, if $Z = 1$.

GENERALIZATION

Nonideal plasmas exhibit frequently not only a high degree of single ionization but also multiple ionization, due to lowering of the ionization energies by the internal Coulomb fields, and over lapping of the atomic wave functions at sufficiently high pressures. In an electrically neutral plasma with N species of ions (i) of charge $Z_i e$ and density n_i, the electron density n and entire ion density n(i) are related by

$$n = \sum_{i=1}^{N} Z_i n_i , \qquad n(i) = \sum_{i=1}^{N} n_i \quad . \tag{44}$$

The characteristic interaction radius $\bar{\delta}$ of the Coulomb field of each ion is within the many-component plasma

$$\bar{\delta} = [4\pi n(i)/3]^{-1/3} \quad . \tag{45}$$

Since the probabilities for interaction between the electrons and ions of type i = i, 2,...N are additive, the momentum relaxation time of the electrons is

$$\tau^{-1} = \sum_{i=1}^{N} \tau_i^{-1} \tag{46}$$

where τ_i is the relaxation time of the momentum exchange of the electrons with ions of type i.

Classical Plasma. By Eqs. (1), (46), and (9)-(10), the conductivity of the many-component plasma with Maxwell electron distribution is in the Lorentz approximation

$$\sigma = \frac{3(KT)^{3/2}}{2(2\pi m)^{1/2} e^2 \sum_{i=1}^{N} (n_i/n) Z_i^2 \cdot L} \quad . \tag{47}$$

L is given by Eq. (11) in general, and the approximations (15) and (16) in the limits $\Lambda >> 1$ and $\Lambda << 1$, respectively, where now

$$\Lambda = 8KT/(\hbar^2/m\bar{\delta}^2) \quad . \tag{48}$$

Quantum Plasma. By Eqs. (1), (46), and (27), the conductivity of the many-component plasma with Fermi electron distribution is in the Lorentz approximation

$$\sigma = \frac{3(KT)^{3/2}\, R(n,T)}{8(2m)^{1/2} e^2 \sum_{i=1}^{N} (n_i/n) Z_i^2 \cdot L_Q} \quad . \tag{49}$$

L_Q is given by Eq. (18) in general, and the approximations (38) and (39) in the limits $\Lambda_Q >> 1$ and $\Lambda_Q << 1$, respectively, where now

$$\Lambda_Q = (16/3)KTQ(n,T)/(\hbar^2/m\,\bar{\delta}^2) \quad . \tag{50}$$

For a plasma with only one ion component, we have $N = 1$, $Z_i = Z$, $n_i/n = Z^{-1}$, and Eqs. (47) and (49) reduce to the previous formulas (17) and (37), respectively. In case of the many-compnent plasma, not only the electron density n but also all ion densities n_i have to be known for the evaluation of the conductivity.

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IV. STATISTICAL THERMODYNAMICS OF NONIDEAL PLASMA

Abstract

A quantum statistical theory of the free energy of a nonideal electron-ion plasma is developed for arbitrary interaction parameters $0 < \gamma < \gamma_c$ ($\gamma = Ze^2 n^{1/3}/KT$ is the ratio of mean Coulomb interaction and thermal energies), which takes into account the energy eigenvalues of (i) the thermal translational particle motions, (ii) the random collective electron and ion motions, and (iii) the static Coulomb interaction energy of the electrons and ions in their oscillatory equilibrium positions. From this physical model, the interaction part of the free energy is derived, which consists of a quasi-lattice energy depending on the interaction parameter γ, and the free energies of the quanitized electron and ion oscillations (long range interactions). Depending on the degree of ordering, the Madelung "constant" of the plasma is $\alpha(\alpha) = \bar{\alpha}$ for $\gamma >> 1$, $\alpha(\gamma) \doteq \bar{\alpha}$ for $\gamma > 1$, and $\alpha(\gamma) \propto \gamma^{1/2}$ for $\gamma << 1$, where $\bar{\alpha} \sim 1$ is a constant. The free energy of the high-frequency plasmons (electron oscillations) is shown to be very small for $\gamma > 1$, whereas the free energy of the low-frequency plasmons (ion oscillations) is shown to be significant for $\gamma > 1$, i.e. for proper nonideal conditions. For weakly nonideal plasmas, $\gamma << 1$, both the electron and ion oscillations contribute to the free energy. Thus, novel results are obtained not only for proper nonideal ($\gamma > 1$) but also for weakly nonideal ($\gamma << 1$) plasmas. From the general formula for the free interaction energy ΔF of the plasma for $0 < \gamma < \gamma_c$, simple analytical expressions are derived for ΔF in the limiting cases, $\gamma >> 1$, $\gamma > 1$, and $\gamma << 1$.

INTRODUCTION

In the classical work of Debye and Hueckel on electrolytes, the total Coulomb interaction energy is calculated from the continuum theoretical picture of every ion interacting with its surrounding space charge cloud. Using more sophisticated methods, similar results were obtained for weakly nonideal plasmas ($\gamma << 1$) by Mayer[1] (cluster expansion), Ichikawa[2] (collective variable approach[3]), Vedenov and Larkin[4] (graphical density expansion), and Jackson[5] (hydrodynamic continuum interaction model). Based on different methods and approximations, investigations of moderately ($\gamma \gtrsim 1$) and strongly ($\gamma >> 1$) nonideal plasmas were given by Berlin and Montroll[6], Theimer and Gentry[7], Ecker and Kroell[8], Ebeling, Hoffman and Kelbg[9], and Varobev, Norman and Vilinov[10], respectively.

In spite of differences in the theoretical approaches, the leading terms of the analytical results for proper nonideal plasmas ($\gamma > 1$) give essentially the same formula for the free plasma energy, $\Delta F/NKT = -a\gamma + b\ln\gamma + c$, due to Coulomb interaction, where $\gamma = Ze^2n^{1/3}/KT$ is the ratio of electron - ion interaction energy and thermal energy, and a, b, c are constants depending on the respective approximations and assumptions. The thermodynamic functions of strongly nonideal plasmas ($\gamma >> 1$) were also determined with the help of Monte Carlo and computer methods by Brush, Sahlin, and Teller[11], Hansen[12], DeWitt[13], and Theimer[14], respectively. Although computer methods provide limited physical insight, they are useful for checking the quantitative validity of analytical theories.

At sufficiently high electron densities, for which $\gamma \geq 1$, classical statistical theories fail due to thermodynamic instability[15], which is inhibited by quantum mechanics. The classical plasma pressure would collapse for $\gamma > 1$ due to the negative electron-ion interaction energy, whereas in reality the pressure remains positive in a plasma due to the Fermi pressure (exclusion principle) of the electrons. For these reasons, we present herein a quantum-statistical theory for nonideal plasmas based on concepts similar to those used by Debye for solids[16)]. The application of this model to proper nonideal plasmas ($\gamma > 1$) is justified since a plasma exhibits a quasi-crystalline structure for $\gamma > 0.1$ before it undergoes a diffuse transition into a solid, metallic state at a critical value γ_c. The roll of the longitudinal phonos of the Debye theory is assumed by the quanta of the plasma oscillations (plasmons) in the case of the quasi-crystalline plasma. The theory is also applicable to weakly nonideal conditions, since the quasi-lattice energy reduces for weak ordering, $\gamma \ll 1$, to the free interaction energy of weakly nonideal plasmas.

The theory to be presented takes into consideration (i) the energy eigenvalues of the random, collective electron and ion oscillations and (ii) the static Coulomb interaction energy (quasi-lattice energy) of the electrons and ions in their oscillatory equilibrium positions. Thus, all significant long and short range Coulomb interactions are considered. The results are applicable to arbitrary nonideal plasmas, $0 < \gamma < \gamma_c$, where γ_c is the critical ordering parameter at which a phase transition into a solid metallic state occurs.

PHYSICAL FOUNDATIONS

Subject of the theoretical considerations are quasi-homogeneous high-pressure plasmas consisting of electrons of charge $-e$ and density $n = N/V$ and ions of charge $+Ze$ and density $n/Z = N/ZV$, with typical densities in the range $10^{20}\ \mathrm{cm}^{-3} \lesssim n \lesssim 10^{24}\ \mathrm{cm}^{-3}$ and temperatures of the order-of-magnitude $T \sim 10^{3} - 10^{4}\ {}^{\circ}\mathrm{K}$. For these conditions, the Debye radius $D = [4\pi n e^2(1+Z)/KT]^{-1/2}$ is $D = 6.901 \times [T/n(1+Z)]^{1/2} \lesssim 10^{-8}\ \mathrm{cm}$, i.e., is smaller than the atomic dimension and the number of particles in the Debye sphere would be $N_D \sim 4\pi n D^3/3 \lesssim 1$ for $D < 10^{-8}\ \mathrm{cm}$ and $n < 10^{24}\ \mathrm{cm}^{-3}$. It is seen that the concept of Debye shielding completely breaks down, and statistical theories containing the Debye length as a characteristic parameter would be physically meaningless for high density plasmas.

The nonideal behavior of plasmas is determined by the interaction parameter γ, which is the ratio of the Coulomb interaction energy $\sim Ze^2 n^{1/3}$ and thermal energy $\sim KT$,

$$\gamma = Ze^2 n^{1/3}/KT = 1.671 \times 10^{-3} Z n^{1/3}/T. \qquad (1)$$

It follows that $0.5Z \leq \gamma \leq 15Z$ for $10^{20}\ \mathrm{cm}^{-3} \leq n \leq 10^{24}\ \mathrm{cm}^{-3}$ and $T \sim 10^{4}\ {}^{\circ}\mathrm{K}$. For $\gamma \doteq 1$, the nature of the plasma changes from a "thermally expanding" ($\gamma < 1$) to an "electrostatically contracting" ($\gamma > 1$) plasma. For $\gamma > 1$, the collapse of the plasma due to Coulomb attraction between electrons and ions is inhibited by the Fermi pressure of the electrons, i.e. by the quantum mechanical exclusion principle. Thus, in the region $0 < \gamma < \gamma_c$ the plasma undergoes a diffuse transition from a nonideal classical plasma ($\gamma \leq 1$) to a quasi-crystalline plasma ($1 < \gamma < \gamma_c$), with an incomplete ordering comparable to that of a liquid.

An understanding of strongly nonideal plasmas has been attempted via the model of discrete interacting particles in a dense gas(6-14). For the above reasons, however, it appears to be a more reasonable procedure to calculate the thermodynamic functions of proper nonideal plasmas from the picture of collective electron and ion oscillations. In this approach, the free interaction energy is due to the static Coulomb interaction of the electrons and ions in their "equilibrium positions" (Madelung energy) and their oscillation energies about the equilibrium positions (plasmon energies).

Since the plasma volume V contains N electrons and N/Z ions, there exist 3N (high-frequency branch) and 3N/Z (low frequency branch) characteristic frequencies ω_i of longitudinal oscillations. Each plasma oscillator of frequency ω_i can only have the energy $(n_i + \frac{1}{2})\hbar\omega_i$, $n_i = 0,1,2,\ldots$, so that the energy $E\{i\}$ of a plasma state with n_i plasmons of frequency ω_i is

$$E\{i\} = \sum_{\{i\}} n_i \hbar\omega_i \tag{2}$$

where $\{i\}$ designates the entire set of given eigenfrequencies ω_i. Accordingly, the partition function Q of the longitudinal plasma oscillations is

$$Q = \sum_{\{i\}} e^{-E\{i\}/KT} = \prod_{\{i\}} 1/(1-e^{-\hbar\omega_i/KT}). \tag{3}$$

From Q, the thermodynamic functions such as the pressure, internal energy, entropy, etc., are derived in the usual way, e.g., the free energy of the plasmons is

$$\tilde{F} = -KT \ln Q = KT \sum_{\{i\}} \ln(1-e^{-\hbar\omega_i/KT}). \tag{4}$$

In the limit $V\to\infty$, the discrete eigenfrequencies ω_i are replaced by continuous ones, $\omega=\omega(k)$, in accordance with the dispersion law for space charge waves of wave length $\lambda = 2\pi/k$, $0 \le k \le \hat{k}$.

1. Electron Oscillations. The high-frequency branch of the space charge waves is due to longitudinal electron oscillations. Their frequency ω is for classical ($n \ll \tilde{n}$) and completely degenerate ($n \gg \tilde{n}$) electrons given by[17)]

$$\omega^2 = \omega_p^2 \, [1 + (r_e/4\pi) Z\gamma^{-1} (k\bar{r}_e)^2] \quad , \quad n \ll \tilde{n} \quad , \tag{5}$$

$$\omega^2 = \omega_p^2 \left[1 + \frac{9}{20\pi}\left(\frac{\pi}{6}\right)^{1/3}\left(\frac{n}{\tilde{n}}\right)^{2/3} Z\gamma^{-1}(k\bar{r}_e)^2\right], \quad n \gg \tilde{n} \quad , \tag{6}$$

where

$$\tilde{n} = 2(2\pi mKT/h^2)^{3/2} \quad , \tag{7}$$

$$\omega_p = (4\pi ne^2/m)^{1/2} \quad , \tag{8}$$

$$\bar{r}_e = n^{-1/3} \quad , \tag{9}$$

are the critical electron density, the plasma frequency, and the mean electron distance ($\kappa_e = c_p/c_v$ of the electrons, and m is their mass). Since $k_{max} \sim 2\pi/\bar{r}_e$ (oscillations with $\lambda < \bar{r}_e$ are physically inconceivable), the electron oscillations propagate, $\omega = \omega(k) > \omega_p$, in nonideal plasmas.

2. Ion Oscillations. The low-frequency branch of the space charge waves is essentially due to ion sound waves. Since the ions are presumed to be nondegenerate, the frequency of the ion oscillations is given by[17)]

$$\omega = \delta(k)(\kappa_i KT/M)^{1/2} k \tag{10}$$

where

$$\delta(k) \doteq \left[1 + \frac{Z(\kappa_e/\kappa_i)}{1 + (\kappa_e/4\pi)Z\gamma^{-1}(k\bar{r}_e)^2}\right]^{1/2} \quad , \quad n \ll \tilde{n} \ , \tag{11}$$

$$\delta(k) \doteq 1 \quad , \quad n \gg \tilde{n} \ , \tag{12}$$

is a correction factor of magnitue-of-order 1, which shows the influence of the electrons on the ion oscillations (M = mass, κ_i = c_p/c_v of the ions).

In weakly nonideal plasmas, $\gamma \ll 1$, the electron sound waves are strongly damped for wave lengths $\lambda < D$, due to trapping of the resonance electrons with thermal speeds comparable to the wave speed. For proper nonideal plasmas, $\gamma > 1$, the number of particles in the Debye sphere $4\pi D^3/3$ is no longer large compared with one and $D < 10^{-8}$ cm is smaller than atomic size, so that thermal Landau damping is no longer feasable. For this reason, electron oscillations should exist for wave length $\lambda > \bar{r}_e$ if $\bar{r}_e > D$.

STATISTICAL THERMODYNAMICS

In the plasma under consideration, the electrons and ions interact through their longitudinal Coulomb fields (transverse electromagnetic interactions are negligible for $KT \ll mc^2$). The electrons ($s = e$) and ions ($s = i$) have thermal velocities $\vec{c}_s$ and random collective mean mass velocities $\tilde{\vec{v}}_s$ due to their oscillatory wave motions about the equilibrium positions, so that $\iiint \vec{c}_s \, f_s \, d^3\vec{c}_s = \vec{0}$, and $\langle \tilde{\vec{v}}_s \rangle = \vec{0}$ where f_s is the equilibrium velocity distribution of the species s and $\langle \; \rangle$ designates a spatial or temporal average. The resulting Hamilton function with Coulomb interaction leads to a free energy of the plasma of the form:

$$F = \sum_{s=e,i} F_s^{(o)} + E_M + \sum_{s=e,i} \tilde{F}_s \quad . \tag{13}$$

$F_s^{(o)}$ is the ideal free energy of the noninteracting plasma components s. E_M is the Coulomb interaction energy of the electrons and ions in their equilibrium positions. $\tilde{F}_{e,i}$ is the free energy of the electron and ion oscillations, i.e. of the high and low frequency plasmons, Eq. (4).

It should be noted that Eq. (9) takes into consideration all significant short-range and long-range Coulomb interactions by means of the Madelung energy E_M and the plasmon energies $\tilde{F}_s$. As is evident from the derivation [17] of Eqs. (5)-(6) and (10), in which terms of order m/M are neglected compared to 1, Eq. (9) contains the e-e, e-i, and i-i Coulomb interactions at distances $\lambda > n^{-1/3}$.

1. Free Energy $F_s^{(o)}$. In high pressure plasmas, the electrons are partially degenerate for densities $n > n_o$ where $n_o = 4.828 \times 10^{15} T^{3/2}$ [cm^{-3}], whereas the ions behave in general classically. Fermi statistics gives for the free energy of the ideal electron gas [18]

$$F_e^{(o)} = -\, NKT\, U_{3/2}(\mu/KT)/U_{1/2}(\mu/KT) \tag{14}$$

where

$$U_\rho(\mu/KT) = \frac{1}{\Gamma(\rho+1)} \int_0^\infty \frac{x^\rho\, dx}{e^{x-\mu/KT}+1} \quad , \rho = 1/2,\ 3/2 \tag{15}$$

and

$$n = 2(2\pi mKT/h^2)^{3/2} \quad U_{1/2}(\mu/KT) \tag{16}$$

defines the Sommerfeld integrals [19], and determines the chemical potential $\mu = \mu(n, T)$ of the electrons, respectively. The free energy of the translational degrees of freedom of the classical, ideal ion gas is [18]

$$F_i^{(o)} = -\, (N/Z)\ KT\ \ln[(2\pi MKT/h^2)^{3/2} \left(\frac{Z}{n}\right)]. \tag{17}$$

2. Quasi-Lattice Energy E_M. The equilibrium positions of the electrons and ions, about which the electrostatic oscillations occur, form an electron "lattice" and an ion "lattice", with an incomplete ordering. By means of the Ewald method [18], one calculates the Coulomb ineraction energy of the quasi-cubically centered electron-ion lattices as

$$E_M = -\, \alpha\gamma NKT \quad , \qquad \alpha \doteq \bar{\alpha} = 1.451 \text{ for } \gamma > 1. \tag{18}$$

As the ordering of the plasma increases with γ, $\alpha(\gamma)$ is a weak function of γ such that asymptotically $\alpha = 1.541$ for $\gamma >> 1$. Eq.(18) indicates that $-E_M/N \approx Ze^2/\bar{r}$ is of the order of the average e-i interaction energy. For weak ordering, $\gamma << 1$, it will be shown that $\alpha \sim \gamma^{1/2}$.

3. <u>High-Frequency Contribution F_e.</u> Since the number of longitudinal modes with wave numbers between k and k + dk in volume V is $V4\pi k^2 dk/(2\pi)^3$, Eq. (4) gives for the free energy F_e of the high-frequency electron oscillations of energy $\hbar\omega(k)$

$$F_e/KT(V/2\pi^2) = \int_0^{\hat{k}_e} \ell n\{1-\exp[-\hbar\omega(k)/KT]\}\, k^2 dk \tag{19}$$

where

$$\omega(k) = \omega_p\,(1+a^2k^2)^{1/2} \quad , \tag{20}$$

$$a^2 \equiv c_m^2/\omega_p^2 = (\kappa_e/4\pi)(Z/\gamma)\bar{r}_e^2 \;, \qquad n \ll \tilde{n} \;, \tag{21}$$

$$a^2 \equiv (3/5)\,\nu_F^2/\omega_p^2 = \frac{9}{20\pi}\left(\frac{\pi}{6}\right)^{1/3}\left(\frac{n}{\tilde{n}}\right)^{2/3}(Z/\gamma)\bar{r}_e^{-2} \;, \qquad n \gg \tilde{n} \;, \tag{22}$$

by Eqs. (5)-(6). The speed of sound c_m and the Fermi speed ν_F of the electrons are

$$c_m = (\kappa_e KT/m)^{1/2} \;, \qquad \nu_F = \hbar(3\pi^2 n)^{1/3}/m. \tag{23}$$

The number of modes in $(0, \hat{k}_e)$ and V equals the number 3N of degrees of freedom of the electron gas, i.e.,

$$(2\pi)^{-3}V \int_0^{\hat{k}_e} 4\pi k^2 dk = 3N, \qquad \hat{k}_e = (18\pi^2 n)^{1/3} \;. \tag{24}$$

Integration of Eq. (19) by parts yields, under consideration of $\hat{k}_e^3\, KTV/6\pi^2 = 3NKT$, for the free energy of the high-frequency plasmons:

$$F_e = 3NKT\,\ell n\{1 - \exp[-\frac{\hbar\omega_p}{KT}(1+a^2\hat{k}_e^2)^{1/2}]\} - F(\frac{\hbar\omega_p}{KT}, a\hat{k}_e) \tag{25}$$

where

$$F(\frac{\hbar\omega_p}{KT}, a\hat{k}_e) = \frac{\hbar\omega_p}{KT}(a\hat{k}_e)^{-3}\int_0^{a\hat{k}_e} \frac{x^4(1+x^2)^{-1/2}dx}{e^{(\hbar\omega_p/KT)(1+x^2)^{1/2}} - 1} \tag{26}$$

and

$$\hbar\omega_p/KT = (4\pi)^{1/2}(\lambda_e/\bar{r}_e)^{1/2}(\gamma/Z)^{1/2} \;, \qquad \lambda_e = \hbar/(mKT)^{1/2} \;, \tag{27}$$

$$a\hat{k}_e = \kappa_e^{1/2}(9\pi^{1/2}/4)^{1/3}(Z/\gamma)^{1/2} \,, \quad n \ll \tilde{n} \,, \tag{28}$$

$$a\hat{k}_e = 2^{1/6}\pi^{1/3}(3\sqrt{3}/2\sqrt{5})(n/\tilde{n})^{1/3}(Z/\gamma)^{1/2} \,, \quad n \gg \tilde{n}. \tag{29}$$

By means of the successive substitutions, (i) $x = \sinh\xi$, $dx = \cosh\xi \, d\xi$ and (ii) $\varepsilon = (\hbar\omega_p/KT)\cosh\xi$, $d\varepsilon = (\hbar\omega_p/KT)\sinh\xi \, d\xi$, the integral (26) is transformed to

$$F(\varepsilon_p, a\hat{k}_e) = (a\hat{k}_e\varepsilon_p)^{-3} \int_{\varepsilon_p}^{\hat{\varepsilon}_e} (\varepsilon^2-\varepsilon_p^2)^{3/2}(e^{\varepsilon}-1)^{-1} \, d\varepsilon \tag{30}$$

where

$$\varepsilon_p = \hbar\omega_p/KT, \qquad \hat{\varepsilon}_e = \varepsilon_p \cosh(\operatorname{arcsinh} a\hat{k}_e). \tag{31}$$

Since the leading expression in Eq. (25) is the logarithmic term, it is sufficient to give for $F(\varepsilon_p, a\hat{k}_e)$ the series approximation (Appendix),

$$F(\varepsilon_p, a\hat{k}_e)/2^{3/2}(a\hat{k}_e)^3\varepsilon_p^{-3/2} =$$

$$\sum_{m=1}^{\infty} e^{-m\varepsilon_p} \sum_{n=0}^{\infty} \binom{3/2}{n}(2\varepsilon_p)^{-n} m^{-(\frac{5}{2}+n)} \gamma(\tfrac{5}{2}+n, (\hat{\varepsilon}_e-\varepsilon_p)m), \quad \hat{\varepsilon}_e < 3\varepsilon_p, \tag{32}$$

where

$$\gamma(\tfrac{5}{2}+n, (\hat{\varepsilon}_e-\varepsilon_p)m) = m^{\frac{5}{2}+n} \int_0^{\hat{\varepsilon}_e-\varepsilon_p} u^{\frac{3}{2}+n} e^{-mu} du \tag{33}$$

is the incomplete gamma function[20]. Since in general $\gamma/Z \gtrless 1$ for $\varepsilon_p < \hat{\varepsilon}_e < 3\varepsilon_p$, the expansion (32) is useful where simple approximate relations do not exist.

4. Low Frequency Contribution $\tilde{F}_i$. With the number of modes in the interval dk at k and volume V given by $V4\pi k^2 dk/(2\pi)^3$, Eq. (4) yields for the free energy $\tilde{F}_i$ of the low-frequency ion oscillations of energy $\hbar\omega(k)$

$$\tilde{F}_i/KT(V/2\pi^2) = \int_0^{\hat{k}_i} \ln\{1-\exp[-\hbar\omega(k)/KT]\}\, k^2 dk \tag{34}$$

where

$$\omega(k) = \delta(k) c_M k, \tag{35}$$

$$c_M = (\kappa_i KT/M)^{1/2} \quad . \tag{36}$$

by Eqs. (10) and (12). The number of modes in $(0, \hat{k}_i)$ and V equals the number 3N/Z of degrees of freedom of the ion gas, i.e.,

$$(2\pi)^{-3} V \int_0^{\hat{k}_i} 4\pi k^2 dk = 3N/Z, \qquad \hat{k}_i = (18\pi^2 n/Z)^{1/3} \quad . \tag{37}$$

Partial integration of Eq. (34) gives, under consideration of $\hat{k}_i^3\, KTV/6\pi^2 = 3(N/Z)KT$, for the free energy of the low-frequency plasmons:

$$F_i = 3(N/Z)KT \left(\ln\{1-\exp[-\frac{\hbar c_M}{KT}\, \delta(\hat{k}_i)\hat{k}_i]\} - G(\hat{k}_i) \right) \tag{38}$$

where

$$G(\hat{k}_i) = \frac{\hbar c_M}{KT}\, \hat{k}_i^{-3} \int_0^{\hat{k}_i} \frac{[\delta(k)+k\delta'(k)]k^3 dk}{e^{(\hbar c_M/KT)\delta(k)k} - 1} \quad . \tag{39}$$

Since the dispersion factor $\delta(k)$ is a bounded function varying very little with k such that $1 \le \delta(k) \lesssim (1+Z)^{1/2}$ for $k \in (0, \hat{k}_i)$, $\delta(k)$ can be approximated by an average value $\bar{\delta}$,

$$\delta(k) = \bar{\delta} \cdot 1, \qquad n \lessgtr \tilde{n} \ . \tag{40}$$

Since in addition the logarithmic expression is the dominant term in Eq. (38), the integral (39) is approximated by

$$G(\varepsilon_i) \simeq \hat{\varepsilon}_i^{-3} \int_0^{\hat{\varepsilon}_i} \varepsilon^2 (e^{\varepsilon} - 1)^{-1}\, d\varepsilon \tag{41}$$

where

$$\varepsilon = \hbar c_M \bar{\delta} k/KT, \qquad \hat{\varepsilon}_i = \hbar c_M \bar{\delta} \hat{k}_i/KT. \tag{42}$$

$G(\hat{\varepsilon}_i)$ has the semi-convergent series expansions,[20)]

$$G(\hat{\varepsilon}_i) = \frac{1}{3}[1-\frac{3}{8}\hat{\varepsilon}_i + \frac{1}{20}\hat{\varepsilon}_i^2 +\ldots], \quad \hat{\varepsilon}_i << 1, \tag{43}$$

$$G(\hat{\varepsilon}_i) = \frac{\pi^4}{15}\hat{\varepsilon}_i^{-3} + 0[e^{-\hat{\varepsilon}_i}], \quad \hat{\varepsilon}_i >> 1. \tag{44}$$

This completes the formal mathematical aspects of the theory, the physical implications of which require further elaboration.

APPLICATIONS

For applications of the theory to strongly, intermediate, and weakly nonideal plasmas, it should be noted that the dimensionless parameters γ/Z, $\hbar\omega_p/KT$, $a\hat{k}_e$, and $n/\tilde{n}$ occuring in Eq. (25) for the free energy $\tilde{F}_e$ of the high-frequency plasmons can not be varied independently. Since both γ/Z and $\lambda_e/\bar{r}_e$ increase with increasing n and decreasing T, $\hbar\omega_p/KT \sim (\lambda_e/\bar{r}_e)(\gamma/Z)^{1/2}$ varies over a large n-T region similar to $(\gamma/Z)^1$, Eq. (7). Numerically,

$$\gamma/Z = 1.670\times 10^{-3} n^{1/3}/T, \quad \hbar\omega_p/KT = 4.310\times 10^{-7} n^{1/2}/T, \quad n/\tilde{n} = 2.071\times 10^{-16} nT^{-3/2},$$
$$a\hat{k}_e = 2.219(\gamma/Z)^{-1/2}, \quad n \ll \tilde{n}$$
$$a\hat{k}_e = 1.910(n/\tilde{n})^{1/3}(\gamma/Z)^{-1/2}, \quad n \gg \tilde{n}. \tag{45}$$

E. g., for $T=10^{4\,o}K$, $\gamma/Z \gtrless 1$ if $n \gtrless 10^{21}\,cm^{-3}$ and $\hbar\omega_p/KT \gtrless 1$ if $n \gtrless 5\times 10^{20}\,cm^{-3}$. For $T=10^{3\,o}K$, $\gamma/Z \gtrless 1$ if $n \gtrless 10^{18}\,cm^{-3}$, etc. Thus, for typical conditions of nonideal plasmas γ/Z and $\hbar\omega_p/KT$ are of the same order of magnitude. It is also recognized that in general $n/\tilde{n} \gg 1$ if $\gamma/Z \gg 1$, and $n/\tilde{n} \ll 1$ if $\gamma/Z \ll 1$.

In Eq. (38) for the free energy $\tilde{F}_i$ of the low frequency plasmons, only one characteristic parameter $\hat{\varepsilon}_i$ occurs since $\delta(k) \sim \bar{\delta} \sim 1$. By Eq. (42), this parameter is

$$\hat{\varepsilon}_i = \frac{\hbar c_M \bar{\delta}\hat{k}_i}{KT} = (18\pi^2)^{1/3}\kappa_i^{1/2}\bar{\delta}\,\frac{\lambda_i}{\bar{r}_i} = 2.158\times 10^{-5} Z^{-1/3}\left(\frac{m}{M}\right)^{1/2}\frac{n}{T}^{1/3}\bar{\delta} \ll 1 \tag{46}$$

where

$$\lambda_i = \hbar/(MKT)^{1/2}, \quad \bar{r}_i = (n/Z)^{-1/3}. \tag{47}$$

Accordingly, for typical nonideal plasma conditions, it is $\hat{\varepsilon}_i \ll 1$ since $\lambda_i/\bar{r}_i \ll 1$ (classical ions) although in general $\lambda_e/\bar{r}_e > 1$ (degenerate electrons) for $\gamma/Z > 1$ or $\hbar\omega_p/KT > 1$.

The deviation ΔF of the free energy of a nonideal plasma from ideality is by Eq. (13) due to the quasi-lattice energy F_M and the plasmon energies $\tilde{F}_{e,i}$,

$$\Delta F = F_M + \sum_{s=e,i} \tilde{F}_s . \tag{48}$$

Since the theory of electron oscillations[17] has not yet been developed for arbitrary degrees of degeneracy ($n \lesssim \tilde{n}$), the contributions of the electron iscillations to ΔF in the cases $n \lesssim \tilde{n}$ and $n \gtrsim \tilde{n}$ have to be estimated from the dispersion equations for $n \ll \tilde{n}$ [Eq. (5)] and $n \gg \tilde{n}$ [Eq. (6)], respectively. Fortunately, it turns out that $|\tilde{F}_e| \ll |\Delta F|$ for $\gamma/Z \gtrsim 1$, so that quantitatively relyable approximations for ΔF can be derived.

1. Strongly Nonideal Plasmas. By Eq. (6) the spectrum $\omega(k)$ of electron oscillations extends over a band $\Delta\omega \sim \omega_p$ above the plasma frequency for $\gamma/Z \gg 1$ since $k\bar{r}_e \leq \hat{k}_e \bar{r} \sim 1$ and $(n/\hat{n})^{2/3} Z\gamma^{-1} \sim 1$. Application of the mean value theorem for integrals to Eq. (25) shows that the free energy $\tilde{F}_e$ of the high-frequency plasmons vanishes exponentially for $\varepsilon_p \to \infty$, i.e. $\gamma/Z \to \infty$:

$$\tilde{F}_e/3NKT = \left(\ell n \{ 1 - \exp[-\varepsilon_p (1+a^2\hat{k}_e{}^2)^{1/2}] \} - \frac{\varepsilon_p (a\hat{k}_e)^{-3}}{\exp[\varepsilon_p (1+\tilde{x}^2)^{1/2}]-1} \int_0^{a\hat{k}_e} x^4 (1+x^2)^{-1/2} dx \right) \to 0, \quad \varepsilon_p \to \infty;$$

$$0 \leq \tilde{x} \leq a\hat{k}_e \ . \tag{49}$$

Accordingly, $|\tilde{F}_e|/3NKT \ll 1$ for $\varepsilon_p \gg 1$, i.e., $\gamma/Z \gg 1$. On the other hand, the free energy of the low frequency plasmons is by Eq. (38) for nondegenerate ions

$$\tilde{F}_i \doteq 3(N/Z)KT[\ell n\hat{\varepsilon}_i - (1/3)] =$$
$$3(N/Z)KT\{\ell n\gamma + \ell n[(18\pi^2/Z^4)^{1/3} \frac{(\kappa_i KT/M)^{1/2}}{e^2/\hbar}] - (1/3)\}, \quad \hat{\varepsilon}_i << 1. \quad (50)$$

It is noted that $\gamma/Z >> 1$ is compatible with $\hat{\varepsilon}_i = \hbar c_M k_i \bar{\delta}/KT << 1$ as explained above.

Equations (49) and (50) demonstrate that the contribution of the electron oscillations to the free energy is negligible in strongly nonideal plasmas, $\gamma/Z >> 1$. In this limit, the nonideal part of the free energy is due to the quasi-lattice energy E_M and the ion oscillations,

$$\Delta F/NKT = - \bar{\alpha}\gamma + (3/Z)\ell n\gamma + (3/Z)\ell n(\beta c_M/v_B) - (1/Z), \quad \gamma/Z >> 1, \quad (51)$$

where

$$v_B = e^2/\hbar, \quad \beta = (18\pi^2 Z^{-4})^{1/3} \quad . \quad (52)$$

Note that $\ell n\gamma$ depends on both n and T whereas $\ell n\beta c_M/v_B$ depends only on T, where the Bohr speed is $v_B = 2.118 \times 10^8$ cm/sec $>> c_M = (\kappa_i KT/M)^{1/2}$.

It is remarkable that the electron oscillations contribute little to the free energy compared to the ion oscillations for $\gamma/Z \gg 1$. This result holds even for moderately nonideal conditions, $\gamma/Z > 1$. Thus, we disagree with the formula "$F = n\varepsilon_o + 3NKT\ell n(\hbar\omega_o/KT)$" stated without derivation for nonideal plasmas by Norman and Starostin[21], according to whom "all the vibrations have exactly the same frequency ω_o near the plasma frequency ω_p". The derivation of this formula requires $\hbar\omega(k)/KT \ll 1$ for the electron oscillations, which implies $\gamma/Z \ll 1$, but the latter inequality contradicts their assumption $\omega(k) \cong \omega_o \cong \omega_p$, since the frequency spectrum extends over a large band $\Delta\omega > \omega_p$ above ω_p for $\gamma/Z \ll 1$. For these reasons, the free energy proposed by them is not applicable to proper nonideal plasmas, $\gamma/Z > 1$, nor is it correct for weakly nonideal conditions, $\gamma/Z \ll 1$.

2. <u>Intermediate Nonideal Plasmas.</u> For intermediate nonideal conditions, $1 \lesssim \gamma/Z < 10$, the spectrum $\omega(k)$ of electron oscillations extends over a region $\Delta\omega \sim O[\omega_p]$ above ω_p by Eq. (6) since $(n/\tilde{n})^{2/3}Z\gamma^{-1} < 1$ and $k\bar{r}_e \simeq \hat{k}_e\bar{r}_\epsilon \sim 1$. Also in this case, a relatively simple formula can be devised for the free energy. The logarithmic term in $\tilde{F}_e$, Eq. (25) is negligible compared to that in $\breve{F}_i$, Eq. (38), for $\gamma/Z > 1$ since $\epsilon_p \gg \hbar c_M \delta \hat{k}_i/KT$ for $\gamma/Z > 1$ by Eqs. (45) and (46), respectively. Accordingly, the nonideal part (48) of the free energy is for intermediate nonideal plasmas:

$$\Delta F/NKT = -\bar{\alpha}\gamma + (3/Z)\ell n\gamma + (3/Z)\ell n(\beta c_M/v_B) - (3/Z)G(\hat{\epsilon}_i) - 3F(\epsilon_p, ak_e), \quad \gamma/Z \gtrsim 1. \tag{53}$$

For $\gamma/Z > 1$, the ions can be assumed to be non-degenerate, $\hat{\epsilon}_i = \hbar c_M \bar{\delta} k_i/KT << 1$ by Eq. (46), so that the ion integral (41) reduces to

$$G(\hat{\epsilon}_i) = 1/3, \quad \hat{\epsilon}_i << 1 . \tag{54}$$

Since $\hat{\epsilon}_e > \epsilon_p \gtrsim 1$ and $\hat{ak}_e \epsilon_p \gtrsim 1$ [Eq. (45)] for $\gamma/Z \gtrsim 1$, the electron integral (30) is significantly smaller than $G(\epsilon_i) = 1/3$,

$$0 < F(\epsilon_p, \hat{ak}_e) < (\hat{\epsilon}_e^2 - \epsilon_p^2)^{3/2} (\hat{ak}_e \epsilon_p)^{-3} \ln(1-e^{-\epsilon_e}/1-e^{-\epsilon_p}) << 1, \quad \gamma/Z \gtrsim 1. \tag{55}$$

The lower and upper bounds of $F(\epsilon_p, \hat{ak}_e)$ have been obtained by means of the mean value theorem for the integral (30),

$$F(\epsilon_p, \hat{ak}_e) = (\hat{ak}_e \epsilon_p)^{-3} (\hat{\epsilon}^2 - \epsilon_p^2)^{3/2} \int_{\epsilon_p}^{\hat{\epsilon}_e} (e^{\epsilon} - 1) d\epsilon , \quad \epsilon_p \leq \hat{\epsilon} \leq \hat{\epsilon}_e . \tag{56}$$

While for strongly nonideal conditions, the contribution of the electron oscillations to the free energy is completely negligible, this contribution is still insignificant for intermediate nonideal conditions, $\gamma/Z \gtrsim 1$, by Eq. (55). For more exact evaluations, the small term $F(\epsilon_p, \hat{ak}_e)$ in Eq. (53) can be computed from Eq. (30) or (32).

3. <u>Weakly Nonideal Plasmas</u>. Although the theory of weakly nonideal systems is well understood,[1-5] it is interesting to investigate whether the present model for proper nonideal plasmas gives reasonable results in the limit $\gamma/Z << 1$. For $\gamma/Z << 1$ it is $\hat{ak}_e >> 1$ by Eq. (45), and the spectrum $\omega(k)$ of electron oscillations extends over a large region $\Delta\omega >> \omega_p$ above ω_p by Eq. (5). The electron integral becomes for $\hat{ak}_e >> 1$,

$$F(\epsilon_p, \hat{ak}_e) = \epsilon_p (\hat{ak}_e)^{-3} \int_0^{\hat{ak}_e} (e^{\epsilon_p x} - 1)^{-1} x^3 dx , \quad \gamma/Z << 1, \tag{57}$$

i.e.,

$$F(\epsilon_p, \hat{ak}_e) = \tfrac{1}{3}[1 - \frac{3}{8}(\epsilon_p \hat{ak}_e)^1 + \frac{1}{20}(\epsilon_p \hat{ak}_e)^2 - \ldots] , \quad \epsilon_p \hat{ak}_e << 1. \tag{58}$$

Although $\varepsilon_p a\hat{k}_e$ is independent of γ/Z by Eqs. (27) and (28), the expansion (58) is valid since the electrons are certainly nondegenerate, $\lambda_e/\bar{r} \ll 1$ for $\gamma/Z \ll 1$, and

$$\varepsilon_p a\hat{k}_e = (4\pi\kappa_e)^{1/2} (9\pi^{1/2}/4)^{1/3} \lambda_e/\bar{r}_e \ll 1, \quad \lambda_e/\bar{r}_e \ll 1. \tag{59}$$

For nondegenerate ions, the integral (41) is $G(\hat{\varepsilon}_i) = 1/3$ by Eq. (43) since $\hat{\varepsilon}_i \ll 1$. Thus, one obtains from Eqs. (18), (25) and (38) for the interaction part of the free energy of weakly nonideal plasmas:

$$\Delta F/NKT = -\tilde{\alpha}(\gamma)\gamma + (3/Z)\ell n\gamma + (3/Z)\ell n(\beta c_M/v_B) + 3\ell n(\varepsilon_p a\hat{k}_e) - (1+Z^{-1}), \quad \gamma/Z \ll 1, \tag{60}$$

where the logarithmic term in Eq. (25) has been expanded for $\varepsilon_p a\hat{k}_e \ll 1$.

In Eq. (60), $\tilde{\alpha}(\gamma)$ is the Madelung constant of the weakly nonideal plasma with weak electron and ion ordering, $\tilde{\alpha}(\gamma) \to 0$ for $\gamma \to 0$. Comparison of the term $-\tilde{\alpha}(\gamma)\gamma(NKT)$ in Eq. (60) with $\Delta F = -(NKT)(2/3)\pi^{1/2}(1+Z)^{3/2}e^3 n^{1/2}(KT)^{-3/2}$ of the Debye-Hueckel theory[23] (weakly nonideal plasmas) yields the result

$$\tilde{\alpha}(\gamma) = (2/3)\pi^{1/2} (1+Z^{-1})^{3/2} \gamma^{1/2}, \qquad \gamma/Z \ll 1. \tag{61}$$

The previous theories of weakly nonideal plasmas do not lead to the logarithmic terms in Eq. (60) since they do not take into account the effects of electron and ion oscillations.

The presented theory is applicable to nonideal plasmas in the gaseous phase ($0<\gamma<1$) and the quasi-liquid ($1<\gamma<\gamma_c$) phase. Whether the Eqs. (25) and (38) are applicable to hot ($T>10^3\,^\circ K$) plasmas in the solid phase ($\gamma>\gamma_c$) cannot be judged at this time, since not enough is experimentally known about the latter, extreme state of matter.

APPENDIX: Expansion of $F(\varepsilon_p, a\hat{k}_e)$

The integral (23) is conveniently rewritten in the form

$$F(\varepsilon_p, a\hat{k}_e) = (a\hat{k}_e \varepsilon_p)^{-3}\, I(\varepsilon_p, \hat{\varepsilon}) \tag{A1}$$

where

$$I(\varepsilon_p, \hat{\varepsilon}) = \int_{\varepsilon_p}^{\hat{\varepsilon}} (\varepsilon^2 - \varepsilon_p^2)^{3/2} (e^{\varepsilon} - 1)^{-1}\, d\varepsilon, \quad 0 < \varepsilon_p < \varepsilon < \infty. \tag{A2}$$

Since $\varepsilon > 0$, i.e. $e^{-\varepsilon} < 1$, there exists the series expansion,

$$(e^{\varepsilon} - 1)^{-1} = \sum_{m=1}^{\infty} e^{-m\varepsilon}, \qquad \varepsilon > 0 . \tag{A3}$$

The substitution, $u = \varepsilon - \varepsilon_p$, $du = d\varepsilon$, and Eq. (A3) transform Eq. (A2) to

$$I(\varepsilon_p, \hat{\varepsilon}) = \sum_{m=1}^{\infty} e^{-m\varepsilon_p} \int_{u=o}^{\hat{\varepsilon} - \varepsilon_p} u^{3/2} (u + 2\varepsilon_p)^{3/2}\, e^{-mu}\, du. \tag{A4}$$

For $u < 2\varepsilon_p$, i.e., $\hat{\varepsilon} < 3\varepsilon_p$, the binomial expansion,

$$(u + 2\varepsilon_p)^{3/2} = (2\varepsilon_p)^{3/2} \sum_{n=0}^{\infty} \binom{3/2}{n} \left(\frac{u}{2\varepsilon_p}\right)^n, \quad u/2\varepsilon_p < 1, \tag{A5}$$

is used, which reduces Eq. (A4) to the double series:

$$I(\varepsilon_p, \hat{\varepsilon}) = (2\varepsilon_p)^{3/2} \sum_{m=1}^{\infty} e^{-m\varepsilon_p} \sum_{n=0}^{\infty} \binom{3/2}{n} (2\varepsilon_p)^{-n}\, m^{-(\frac{5}{2}+n)} \gamma\left(\frac{5}{2} + n, (\hat{\varepsilon} - \varepsilon_p) m\right),$$

$$\hat{\varepsilon} < 3\varepsilon_p, \tag{A6}$$

where

$$\gamma\left(\frac{5}{2} + n, (\hat{\varepsilon} - \varepsilon_p) m\right) = m^{\frac{5}{2}+n} \int_0^{\hat{\varepsilon} - \varepsilon_p} u^{\frac{3}{2}+n} e^{-mu}\, du. \tag{A7}$$

is the incomplete gamma function, which is tabulated.[20] In an amologous way, the integral (A2) can be solved for $u > 2\varepsilon_p$, i.e., $3\varepsilon_p < \hat{\varepsilon} < \infty$.

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V. VARIATIONAL PRINCIPLES AND CANONICAL THEORY FOR MANY - COMPONENT PLASMAS WITH SELF-CONSISTENT FIELD

ABSTRACT

The Lagrangian $L = \iiint L(\vec{q}_s, \partial\vec{q}_s/\partial t, \nabla\cdot\vec{q}_s)d^3\vec{r}$ and Hamiltonian $H = \iiint H(\vec{p}_s, \vec{q}_s, \nabla\cdot\vec{q}_s)d^3\vec{r}$ for many-component plasmas with self-consistent field interactions are derived. The canonical fields $\vec{p}_s(\vec{r},t)$ and $\vec{q}_s(\vec{r},t)$ are defined by $\vec{p}_s = m_s\vec{v}_s$ and $\partial\vec{q}_s/\partial t = n_s\vec{v}_s$, respectively, where $n_s(\vec{r},t)$ is the density and $\vec{v}_s(\vec{r},t)$ is the velocity of the component s, $s = 1,2,\ldots N$. Based on the action principle, the Lagrange and Hamilton equations of motion for the components (s) are presented as functional derivatives of L and H with respect to the canonical momenta $\vec{p}_s$ and $\vec{q}_s$. It is shown that the new formulations of many-component plasma dynamics have mathematical advantages compared with the conventional variational principles and hydrodynamic equations.

INTRODUCTION

Since the work of Clebsch [1] it is known that the space and time dependent dynamics of perfect fluids can be derived from variational principles. [2-5] To-date, however, no variational principle has been found the extremals of which give the Navier-Stokes equations with viscous dissipation. A comprehensive review of the most significant contributions to variational principles in fluid mechanics has been presented by Serrin. [6] Similar variational principles for perfect one-fluid plasmas and magnetohydrodynamic fluids have been given by Newcomb [7] and Zakharov. [8]

Herein, a new variational principle and canonical theory for perfect, non-relativistic many-component plasmas is developed, in which the charged particles interact through the self-consistent electric field. Interactions through self-consistent magnetic fields are neglected for non-relativistic velocities $v_e << c$ and temperatures $kT_e << m_e c^2$ of the electrons. We show that the electro-hydrodynamic equations of the components s = 1, 2, ... N of the plasma are derivable as functional derivatives [9] of the Lagrange function $L = \iiint_{\Omega} \mathcal{L}(\vec{r},t)d^3\vec{r}$ and Hamilton function $H = \iiint_{\Omega} \mathcal{H}(\vec{r},t)d^3\vec{r}$ of the plasma of volume Ω. The Lagrange $\mathcal{L}$ and Hamilton $\mathcal{H}$ densities are functionals of the canonical congugate variables $\vec{p}_s$ and $\vec{q}_s$ defined by $\vec{p}_s = m_s\vec{v}_s(\vec{r},t)$ and $\partial\vec{q}_s/\partial t = n_s(\vec{r},t)\vec{v}_s(\vec{r},t)$, where m_s is the mass, n_s is the density, and $\vec{v}_s$ is the (local) average velocity of the particles of the s-th component.

The variational principle and canonical theory for many-component plasmas provides a theoretical foundation for the analysis of classical phenomena such as nonlinear electron and ion acoustic waves, space charge fluctuations, electrostatic turbulence, and electrohydrodynamic initial-boundary-value problems. The Hamilton function and canonical formalism were derived as a theoretical basis for a statistical mechanics of hydrodynamic plasma fields concerned with the evaluation of the free energy of electron-ion plasmas, which are nonideal due to the self-consistent field interactions. The statistics of fields and wave modes represents an alternative to the treatment of many-body systems within the framework of the discrete particle picture.

VLASOV PLASMA

The subject of consideration is a plasma consisting of an electron component and N-1 ion components, which interact through the self-consistent electric field $\vec{E}(\vec{r},t)$ of all charged particles. The hydrodynamic equations for each component s of the plasma are obtained as moments of the Vlasov equation for its velocity distribution function $f_s(\vec{v},\vec{r},t)$, and are in absence of collisions (s=1,2,...N):[10]

$$\frac{\partial}{\partial t}(n_s m_s \vec{v}_s) + \nabla\cdot(n_s m_s \vec{v}_s \vec{v}_s) = -\nabla P_s + n_s e_s \vec{E}, \tag{1}$$

$$\frac{\partial}{\partial t} n_s + \nabla\cdot(n_s \vec{v}_s) = 0 \quad , \tag{2}$$

$$\frac{\partial}{\partial t} P_s + \nabla\cdot(P_s \vec{v}_s) = -(\gamma_s - 1) P_s \nabla\cdot\vec{v}_s \quad , \tag{3}$$

where

$$\nabla\cdot\vec{E} = \varepsilon_o^{-1} \sum_{s=1}^{N} n_s e_s \tag{4}$$

relates the self-consistent field $\vec{E}(\vec{r},t)$ to its space charge sources $n_s(\vec{r},t)e_s$. (ε_o is the dielectric permittivity). The charge of the particles is $e_s = -e$ for the electrons and $e_s = Z_s e$, $|Z_s| = 1,2,3,\ldots$, for the ions (e is the elementary charge). The moments are the density, velocity, and pressure fields of the s-th component,

$$n_s(\vec{r},t) = \iiint_{-\infty}^{+\infty} f_s(\vec{v},\vec{r},t)\, d^3\vec{v} \quad ,$$

$$\vec{v}_s(\vec{r},t) = n_s^{-1} \iiint_{-\infty}^{+\infty} \vec{v} f_s(\vec{v},\vec{r},t)\, d^3\vec{v} \quad ,$$

$$P_s(\vec{r},t) = (2/3) \iiint_{-\infty}^{+\infty} \tfrac{1}{2} m_s (\vec{v} - \vec{v}_s)^2 f_s(\vec{v},\vec{r},t)\, d^3\vec{v},$$

Furthermore, m_s is the mass of the s particles and $\gamma_s = c_s^P/c_s^V$ is the adiabatic coefficient of the s component ($\vec{v}$ is the particle velocity).

Equations (1) - (4) describe the macroscopic dynamics of an N-component Vlasov plasma,[10] in which external magnetic fields are assumed to be absent, and internal magnetic fields and binary correlations are negligible. Maxwell's equations indicate that in the absence of magnetic fields, $\vec{H} = \vec{0}$:

$$\nabla \times \vec{E} = \vec{0} \quad , \tag{5}$$

$$\varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \sum_{s=1}^{N} n_s e_s \vec{v}_s = \vec{0} \quad . \tag{6}$$

Thus, the self-consistent field $\vec{E}(\vec{r},t)$ is longitudinal, and the displacement $(\varepsilon_0 \partial\vec{E}/\partial t)$ and convection $(n_s e_s \vec{v}_s)$ currents annihilate themselves at every point $(\vec{r},t)$. This effect was first noted by Ehrenfest in his discussion of "accelerated charge motions without electromagnetic radiation". [11] Typical examples for accelerated charge motions without transverse radiation field are electron and ion acoustic waves, and space charge turbulence.

Substitution of Eq. (6) into Eq. (4) yields the conservation equation, $\partial \sum n_s e_s/\partial t + \nabla\cdot\sum n_s e_s \vec{v}_s = 0$, for the space charge. The latter equation is, therefore, not independent [in addition, it is obtainable by summing Eq. (2) multiplied by e_s with respect to s].

Due to the absence of i) external magnetic fields and ii) binary interactions (vanishing viscous and thermal dissipation), the pressure in Eqs. (1) and (3) is isotropic. The closure of the Eqs. (1)-(4) is accomplished by setting all higher order moments of the velocity distribution zero, such as the off-diagonal stress components, $\Pi_{ij} \equiv 0$, the third order heat flux tensor, $q_{ijk} \equiv 0$, etc. The resulting electrohydrodynamic or hydrodynamic Vlasov equations are generally accepted in the literature.

EXTREMUM PRINCIPLE

The dynamics of the many-component plasma is now condensed into a variational principle for the Lagrange integral functional $L = \iiint_{\Omega} L(\vec{r},t)d^3\vec{r}$, the extremals of which give the Eulerian equations (1) for the components s = 1,2,...N. In this approach, Eqs. (2) - (4) are auxiliary equations by means of which the fields $n_s(\vec{r},t)$, $P_s(\vec{r},t)$, and $\vec{E}(\vec{r},t)$ can be eliminated from Eq. (1). This elimination will be carried through after introducing canonical variables.

In analogy to the Lagrange density $L(\vec{r},t)$ for nonconducting fluids,[5] the Lagrange density for the many-component plasma with the self-consistent field $\vec{E}(\vec{r},t)$ is sought in the form

$$L(\vec{r},t) = \sum_{s=1}^{N} (\tfrac{1}{2} n_s m_s \vec{v}_s^{\,2} - \frac{P_s}{\gamma_s - 1}) - \frac{\epsilon_o}{2} \vec{E}^2 \quad . \tag{7}$$

If Eq. (7) is the correct Lagrange density, the Eulerian equations of motion for the plasma components should follow from the extremum condition [6]

$$\delta \int_{t_1}^{t_2} dt \iiint L(\vec{r},t) d^3\vec{r} = 0, \tag{8}$$

i.e.,

$$\int_{t_1}^{t_2} dt \iiint_{\Omega} \sum_{s=1}^{N} [n_s m_s \vec{v}_s \cdot \delta \vec{v}_s + \frac{1}{2} m_s \vec{v}_s^{2} \delta n_s - \frac{\delta P_s}{\gamma_s - 1} - \epsilon_o \vec{E} \cdot \delta \vec{E}] d^3\vec{r} = 0. \tag{9}$$

The Eulerian virtual displacement, $\vec{\sigma}_s(\vec{r},t)$, of a mass element of the s-th component,

$$\vec{\sigma}_s(\vec{r},t) = \vec{v}_s(\vec{r},t)\delta t, \tag{10}$$

vanishes at the initial (t_1) and final (t_2) instants of the variation of the trajectory of the fluid element which occurs in the interval $t_1 < t < t_2$,

$$\vec{\sigma}_s(r,t_1) = \vec{0}, \quad \vec{\sigma}_s(\vec{r},t_2) = \vec{0}, \tag{11}$$

and is tangential to the planes $d\vec{A}$ bounding the plasma at r = s:

$$\sigma_s(\vec{r},t)_{r=s} \cdot d\vec{A}(\vec{s}) = 0, \tag{12}$$

The variations of the fields of each component s are given as functionals of its displacement $\vec{\sigma}_s(\vec{r},t)$:

$$\delta\vec{v}_s = \frac{\partial\vec{\sigma}_s}{\partial t} + \vec{v}_s\cdot\nabla\vec{\sigma}_s - \vec{\sigma}_s\cdot\nabla\vec{v}_s, \tag{13}$$

$$\delta n_s = -\nabla.(n_s\vec{\sigma}_s), \tag{14}$$

$$\delta P_s = -\vec{\sigma}_s\cdot\nabla P - \gamma_s P_s \nabla\cdot\vec{\sigma}_s, \tag{15}$$

$$\delta\vec{E} = -\varepsilon_o^{-1}\sum_{s=1}^{N} n_s e_s \vec{\sigma}_s \quad . \tag{16}$$

Equation (13) is obtained by setting the variation of the Lagrangian velocity equal to the substantial derivative of the Eulerian displacement. Equations (13)-(16) result directly from Eqs. (2), (3), and (6), by means of Eq. (10). Substitution of Eqs. (14)-(15) into Eq. (9) gives

$$\int_{t_1}^{t_2} dt \iiint_{\Omega} \sum_{s=1}^{N} [n_s m_s \vec{v}_s\cdot(\frac{\partial\vec{\sigma}_s}{\partial t} + \vec{v}_s\cdot\nabla\vec{\sigma}_s - \vec{\sigma}_s\cdot\nabla\vec{v}_s) - \frac{1}{2} m_s \vec{v}_s^2 \nabla\cdot(n_s\vec{\sigma}_s)$$

$$+ \frac{1}{\gamma_s - 1}(\vec{\sigma}_s\cdot\nabla P_s + \gamma_s P_s \nabla\cdot\vec{\sigma}_s) + n_s e_s \vec{E}\cdot\vec{\sigma}_s] d^3\vec{r} = 0 \quad . \tag{17}$$

By means of partial integrations, Gauss' integral theorem, and the surface condition (12), Eq. (17) is reduced to

$$\int_{t_1}^{t_2} dt \iiint_{\Omega} \sum_{s=1}^{N} \vec{\sigma}_s\cdot[\frac{\partial}{\partial t}(n_s m_s \vec{v}_s) + \nabla\cdot(n_s m_s \vec{v}_s \vec{v}_s) + \nabla P_s - n_s e_s \vec{E}] d^3\vec{r} = 0 \quad . \tag{18}$$

Since Eq. (18) is zero for arbitrary variations $\vec{\sigma}_s$, s = 1,2,...N, the integrand in the bracket must vanish at any point $(\vec{r},t)$ of the plasma. Thus, the Eulerian equations of motion (1) are obtained for the N components from the variational principle (8). This completes the proof that Eq. (7) is the correct Lagrange density for the multicomponent plasma.

The Lagrange density L and the Hamilton density H are not only functionals of fields of $\vec{r}$ and t, but also depend, in general, explicitly on $\vec{r}$ and t. In the following canonical theory, the explicit $\vec{r}$,t dependence need not to be formally indicated, since the virtual displacements constructing the varied paths are for constant $\vec{r}$ and t.

LAGRANGE EQUATIONS

In the Lagrangian formulation of many-component plasma dynamics, the Lagrangian $L = \iiint \mathcal{L}(\vec{r},t)d^3\vec{r}$ is expressed as a functional of the fields $\vec{q}_s(\vec{r},t)$, which will be shown to form with the fields $\vec{p}_s(\vec{r},t)$ canonical variables. These are defined by, s = 1,2,...N:

$$\vec{p}_s = m_s\vec{v}_s, \quad \partial\vec{q}_s/\partial t = n_s\vec{v}_s \quad . \tag{19}$$

Note that $\vec{p}_s$ is proportional to the velocity field $\vec{v}_s$, whereas $\vec{q}_s$ is a t-integral functional of the particle flux $n_s\vec{v}_s$. Integration of the auxilary equations (12), (3), (6) [Eq. (6) is equivalent to Eq. (4)], and Eq. (19) yield

$$n_s = \nu_s - \nabla\cdot\vec{q}_s, \quad \nu_s = \nu_s(\vec{r}) \quad , \tag{20}$$

$$P_s = P_{so}[(\nu_s - \nabla\cdot\vec{q}_s)/n_{so}]^{\gamma_s} \quad , \tag{21}$$

$$\vec{E} = \vec{\varepsilon}_s - \varepsilon_o^{-1}e_s\vec{q}_s, \quad \vec{\varepsilon}_s = \vec{\varepsilon}_s(\vec{r}), \quad , \tag{22}$$

$$\vec{v}_s = (\partial\vec{q}_s/\partial t)/(\nu_s - \nabla\cdot\vec{q}_s) \quad . \tag{23}$$

P_{so}, n_{so}, ν_s, and $\vec{\varepsilon}_s$ are constants with respect to the integration variables. Substitution of Eqs. (20) - (23) into Eq. (7) gives the Lagrange density of the plasma as a functional of the $\vec{q}_s$,

$$\mathcal{L} = \sum_{s=1}^{N} \mathcal{L}_s(\vec{q}_s, \partial\vec{q}_s/\partial t, \nabla\cdot\vec{q}_s) \tag{24}$$

where

$$\mathcal{L}_s = \tfrac{1}{2}m_s \frac{(\partial\vec{q}_s/\partial t)^2}{\nu_s - \nabla\cdot\vec{q}_s} - \frac{P_{so}}{\gamma_s - 1}\left(\frac{\nu_s - \nabla\cdot\vec{q}_s}{n_{so}}\right)^{\gamma_s} - \frac{\varepsilon_o}{2}(\vec{\varepsilon}_s - \varepsilon_o^{-1}e_s\vec{q}_s)^2 \tag{25}$$

is the Lagrange density of the s component. The Lagrangians of the plasma and its components are given by

$$L = \sum_{s=1}^{N} L_s, \quad L_s = \iiint_{\Omega} \mathcal{L}_s d^3\vec{r}. \tag{26}$$

The fields $\vec{p}_s$ and $\vec{q}_s$ defined in Eq. (19) are canonical variables since by Eqs. (24) - (25)

$$\vec{p}_s = \frac{\partial \mathcal{L}}{\partial(\partial\vec{q}_s/\partial t)} = \frac{\partial\vec{q}_s/\partial t}{\nu_s - \nabla\cdot\vec{q}_s} = m_s\vec{v}_s . \tag{27}$$

Thus, by introducing the canonical variable $\vec{q}_s$, $\mathcal{L}$ has been decomposed into the $\mathcal{L}_s$ of the individual components. Note that the original Lagrange density in Eq. (7) does not permit such a decomposition due to the coupling of the components by the common, nonlinear field energy $\varepsilon_o\vec{E}^2/2$.

For the Lagrange density of the form of Eq. (24), the variational principle in Eq. (8),

$$\delta \int_{t_1}^{t_2} dt \iiint_{\Omega} \mathcal{L} d^3\vec{r} = \sum_{s=1}^{N} \delta \int_{t_1}^{t_2} dt \iiint_{\Omega} \mathcal{L}_s d^3\vec{r} = 0, \tag{28}$$

becomes, since the $\mathcal{L}_s$ are independent for $s = 1,2,\ldots N$,

$$\delta \int_{t_1}^{t_2} dt \iiint_{\Omega} \mathcal{L}_s(\vec{q}_s, \partial\vec{q}_s/\partial t, \nabla\cdot\vec{q}_s) d^3\vec{r} = 0. \tag{29}$$

Hence,

$$\int_{t_1}^{t_2} dt \iiint_{\Omega} [\frac{\partial \mathcal{L}_s}{\partial \vec{q}_s}\cdot\delta\vec{q}_s + \frac{\partial \mathcal{L}_s}{\partial(\partial\vec{q}_s/\partial t} \cdot \delta(\frac{\partial\vec{q}_s}{\partial t}) + \frac{\partial \mathcal{L}_s}{\partial\nabla\cdot\vec{q}_s} \delta\nabla\cdot\vec{q}_s] d^3\vec{r} = 0 \tag{30}$$

where

$$\delta\vec{q}_s = (\partial\vec{q}_s/\partial t)\delta t = n_s\vec{v}_s\delta t = n_s\vec{\sigma}_s, \tag{31}$$

$$\delta\vec{q}_s(\vec{r},t_1) = \vec{0}, \quad \delta\vec{q}_s(\vec{r},t_2) = \vec{0}, \quad \delta\vec{q}_s(r,t)_{r=s}\cdot d\vec{A}(s) = 0, \tag{32}$$

by Eqs. (10) - (12). By means of the partial integrations,

$$\int_{t_1}^{t_2} \frac{\partial L_s}{\partial(\partial\vec{q}_s/\partial t)} \cdot \delta\left(\frac{\partial\vec{q}_s}{\partial t}\right) dt = \left[\frac{\partial L_s}{\partial(\partial\vec{q}_s/\partial t)} \cdot \delta\vec{q}_s\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \partial\vec{q}_s \cdot \frac{\partial}{\partial t} \frac{\partial L_s}{\partial(\partial\vec{q}_s/\partial t)} dt, \tag{33}$$

$$\iiint_\Omega \frac{\partial L_s}{\partial\nabla\cdot\vec{q}_s} \delta\nabla\cdot\vec{q}_s d^3\vec{r} = \oint \frac{\partial L_s}{\partial\nabla\cdot\vec{q}_s} \delta\vec{q}_s \cdot d\vec{A} - \iiint_\Omega \partial\vec{q}_s \cdot \nabla\left(\frac{\partial L_s}{\partial\nabla\cdot\vec{q}_s}\right) d^3\vec{r}, \tag{34}$$

and Eq. (32), the extremum principle is reduced to

$$\int_{t_1}^{t_2} dt \iiint \left[\frac{\partial L_s}{\partial\vec{q}_s} - \frac{\partial}{\partial t} \frac{\partial L_s}{\partial(\partial\vec{q}_s \partial t)} - \nabla\left(\frac{\partial L_s}{\partial\nabla\cdot\vec{q}_s}\right)\right] \cdot \delta\vec{q}_s d^3\vec{r} = 0 \quad . \tag{35}$$

The condition for the vanishing of this integral for arbitrary $\delta\vec{q}_s$ leads to the Lagrangian equation of motion of the s component of the plasma:

$$\frac{\partial}{\partial t} \frac{\partial L_s}{\partial(\partial\vec{q}_s/\partial t)} - \frac{\partial L_s}{\partial\vec{q}_s} + \nabla\left(\frac{\partial L_s}{\partial\nabla\cdot\vec{q}_s}\right) = \vec{w}_s \quad , \tag{36}$$

where

$$\vec{w}_s = m_s\left(\frac{\partial\vec{q}_s/\partial t}{\nu_s - \nabla\cdot\vec{q}_s}\right) \times \nabla \times \left(\frac{\partial\vec{q}_s/\partial t}{\nu_s - \nabla\cdot\vec{q}_s}\right) \tag{37}$$

is a vector field perpendicular to $\delta\vec{q}_s$, $\vec{w}_s \cdot \delta\vec{q}_s = 0$ by Eq. (31). It is noted that the extremals of Eq. (35) are undetermined up to fields $\vec{w}_s \perp \delta\vec{q}_s$ ($\vec{w}_s \simeq m_s\vec{v}_s \times \nabla \times \vec{v}_s \neq \vec{0}$ for $\vec{v}_s \neq -\nabla\psi_s$).

In terms of the Lagrangian of the multi-component plasma, Eq. (26), and functional derivatives,[9] the Lagrange equation (36) assumes the most general form,

$$\frac{\partial}{\partial t} \frac{\delta L}{\delta(\partial\vec{q}_s/\partial t} - \frac{\delta L}{\delta\vec{q}_s} = \vec{w}_s, \tag{38}$$

where

$$\frac{\delta L}{\delta\vec{q}_s} = \frac{\partial L_s}{\partial\vec{q}_s} - \nabla\left(\frac{\partial L_s}{\partial\nabla\cdot\vec{q}_s}\right), \qquad \frac{\delta L}{\delta(\partial\vec{q}_s/\partial t)} = \frac{\partial L_s}{\partial(\partial\vec{q}_s/\partial t)} . \tag{39}$$

Substitution of L_s from Eq. (25) into Eq. (36) indeed yields the Eulerian equation (1), since $\frac{1}{2}\nabla(\vec{v}_s{}^2) - \vec{v}_s \times \nabla \times \vec{v}_s = \vec{v}_s \cdot \nabla \vec{v}_s$

HAMILTON EQUATIONS

Another important formulation of the dynamics of a many-component plasma is obtained by means of the Hamilton formalism. Since $\vec{p}_s$ and $\vec{q}_s$ are canonical momenta, the Hamilton density of a component s is

$$H_s = \vec{p}_s \cdot \frac{\partial \vec{q}_s}{\partial t} - L_s, \tag{40}$$

i.e.,

$$H_s = \frac{1}{2m_s} (\nu_s - \nabla \cdot \vec{q}_s) \vec{p}_s^{\,2} + \frac{p_{so}}{\gamma_s - 1} \left(\frac{\nu_s - \nabla \cdot \vec{q}_s}{n_{so}}\right)^{\gamma_s} + \frac{\varepsilon_o}{2} (\vec{\varepsilon}_s - \varepsilon_o^{-1} e_s \vec{q}_s)^2 \tag{41}$$

by Eq. (25). Accordingly, the Hamilton density of the plasma is a functional of the form

$$H = \sum_{s=1}^{N} H_s(\vec{p}_s, \vec{q}_s, \nabla \cdot \vec{q}_s) . \tag{42}$$

The resulting Hamiltonians of the plasma and its components are

$$H = \sum_{s=1}^{N} H_s, \quad H_s = \iiint_{\Omega} H_s \, d^3\vec{r}. \tag{43}$$

Thus, also within the Hamilton approach, a decomposition of the Hamiltonian H into component Hamiltonians H_s is achieved, each H_s depending only on the fields of the component s. This is quite remarkable, since the various components s interact through the self-consistent field $\vec{E}(\vec{r},t)$. The decomposition is physically possible, since the electric fields of the different components superimpose in a linear way.

The Hamilton equations of motion of the components s follow from the action principle (29),

$$\delta \int_{t_1}^{t_2} dt \iiint_{\Omega} (\vec{p}_s \cdot \frac{\partial \vec{q}_s}{\partial t} - H_s) d^3\vec{r} = 0, \tag{44}$$

which gives, since the Hamilton density is a function of the form

$H_s = H_s(\vec{p}_s, \vec{q}_s, \nabla\cdot\vec{q}_s)$,

$$\int_{t_1}^{t_2} dt \iiint_\Omega [(\frac{\partial\vec{q}_s}{\partial t} - \frac{\partial H_s}{\partial\vec{p}_s})\cdot\delta\vec{p}_s + \vec{p}_s\cdot\delta(\frac{\partial\vec{q}_s}{\partial t}) - \frac{\partial H_s}{\partial\vec{q}_s}\cdot\delta\vec{q}_s - \frac{\partial H_s}{\partial\nabla\cdot\vec{q}_s}\,\delta\nabla\cdot\vec{q}_s]d^3\vec{r} = 0 \quad . \tag{45}$$

Interchange of the sequence of variation and differentiation and partial integrations,

$$\int_{t_1}^{t_2} \vec{p}_s\cdot\frac{\partial}{\partial t}\delta\vec{q}_s dt = [\vec{p}_s\cdot\delta\vec{q}_s]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\partial\vec{p}_s}{\partial t}\cdot\delta\vec{q}_s dt, \tag{46}$$

$$\iiint_\Omega \frac{\partial H_s}{\partial\nabla\cdot\vec{q}_s}\nabla\cdot\delta\vec{q}_s d^3\vec{r} = \oint \frac{\partial H_s}{\partial\nabla\cdot\vec{q}_s}\delta\vec{q}_s\cdot d\vec{A} - \iiint_\Omega \delta\vec{q}_s\cdot\nabla(\frac{\partial H_s}{\partial\nabla\cdot\vec{q}_s})d^3\vec{r}, \tag{47}$$

where the variation $\delta\vec{q}_s$ satisfies the condition in Eq. (32), transform Eq. (45) to

$$\int_{t_1}^{t_2} dt \iiint_\Omega \{(\frac{\partial\vec{q}_s}{\partial t} - \frac{\partial H_s}{\partial\vec{p}_s})\cdot\delta\vec{p}_s - [\frac{\partial\vec{p}_s}{\partial t} + \frac{\partial H_s}{\partial\vec{q}_s} - \nabla(\frac{\partial H_s}{\partial\nabla\cdot\vec{q}_s})]\cdot\delta\vec{q}_s\}d^3\vec{r} = 0. \tag{48}$$

The conditions for the vanishing of this integral for arbitrary variations $\delta\vec{p}_s$ and $\delta\vec{q}_s$ gives the Hamilton equations of motion for the plasma components s:

$$\frac{\partial\vec{q}_s}{\partial t} = \frac{\partial H_s}{\partial\vec{p}_s} \quad , \tag{49}$$

$$\frac{\partial\vec{p}_s}{\partial t} + \frac{\partial H_s}{\partial\vec{q}_s} - \nabla(\frac{\partial H_s}{\partial\nabla\cdot\vec{q}_s}) = \vec{w}_s \quad , \tag{50}$$

where

$$\vec{w}_s = \frac{1}{m_s}\vec{p}_s \times \nabla \times \vec{p}_s \tag{51}$$

is a vector field perpendicular to $\delta\vec{q}_s = n_s\vec{v}_s\delta t$, i.e., $\vec{w}_s\cdot\delta\vec{q}_s = 0$. As above, $\vec{w}_s$ is determined from the condition of rotational flow ($\vec{w}_s \neq \vec{0}$ for $\nabla \times \vec{v}_s \neq \vec{0}$).

By introducing the Hamiltonian of the plasma, Eq. (46), and functional derivatives,[9] the Hamilton equations (49) - (50) are brought into the most general form,

$$\frac{\partial \vec{q}_s}{\partial t} = \frac{\delta H}{\delta \vec{p}_s} \quad , \tag{52}$$

$$\frac{\partial \vec{p}_s}{\partial t} = -\frac{\delta H}{\delta \vec{q}_s} \quad , \tag{53}$$

where

$$\frac{\delta H}{\delta \vec{p}_s} = \frac{\partial H_s}{\partial \vec{p}_s} \, , \qquad \frac{\delta H}{\delta \vec{q}_s} = \frac{\partial H_s}{\partial \vec{q}_s} - \nabla \left(\frac{\partial H_s}{\partial \nabla \cdot \vec{q}_s} \right) \quad . \tag{54}$$

Evaluation of Eqs. (49) - (50) for the Hamilton density in Eq. (41) results in Eq. (19) and the Eulerian equation of motion (1), respectively.

CONCLUSIONS

Comprehensive Lagrangian and Hamiltonian methods have been developed for many-component plasmas with internal interactions through the self-consistent electric field. The theory is based on the Lagrange density (26) and the Hamilton density (41), and the canonical variables $\vec{p}_s$ and $\vec{q}_s$ defined by $\vec{p}_s = m_s \vec{v}_s$ and $\partial \vec{q}_s / \partial t = n_s \vec{v}_s$, respectively. In the limit of a single component (N=1) and vanishing Coulomb interactions, the corresponding Lagrange and Hamilton equations of motion for an ordinary neutral particle gas result.

The canonical theory presented reduces the coupled partial differential equations (1) - (4) for the fields $\vec{v}_s(\vec{r},t)$, $n_s(\vec{r},t)$, $P_s(\vec{r},t)$, and $\vec{E}(\vec{r},t)$ to a single partial differential equation for the canonical field $\vec{q}_s(\vec{r},t)$. The fundamental Lagrange equation of motion is

$$m_s\left[\frac{\partial}{\partial t}\left(\frac{\partial\vec{q}_s/\partial t}{\nu_s-\nabla\cdot\vec{q}_s}\right)+\left(\frac{\partial\vec{q}_s/\partial t}{\nu_s-\nabla\cdot\vec{q}_s}\right)\cdot\nabla\left(\frac{\partial\vec{q}_s/\partial t}{\nu_s-\nabla\cdot\vec{q}_s}\right)\right]$$

$$=-\frac{p_{so}/n_{so}^{\gamma}}{\nu_s-\nabla\cdot\vec{q}_s}\nabla(\nu_s-\nabla\cdot\vec{q}_s)^{\gamma_s}+e_s(\vec{\varepsilon}_s-\varepsilon_o^{-1}e_s\vec{q}_s) \tag{55}$$

by Eqs. (26) and (36). From the solutions $\vec{q}_s(\vec{r},t)$, $s = 1,2,\ldots N$, of Eq. (55), the original plasma fields are obtained by means of Eqs. (20) - (23). The mathematical advantages of the theory presented are obvious, in particular for plasmas with several components. In comparison with the previous formulations of variational principles in fluid mechanics [1,3,4,5] which make use of an excessive number of scalar Clebsch parameters, the present canonical theory in terms of the canonical vector fields $\vec{p}_s$ and $\vec{q}_s$ excells in mathematical simplicity.

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$$\frac{\delta F[\Psi(x)]}{\delta\Psi(x)} = \lim_{\varepsilon\to 0} \frac{F[\Psi(x') + \Psi\,\delta(x - x')] - F[\Psi(x')]}{\varepsilon}$$

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VI. DISTRIBUTION FUNCTION OF TURBULENT VELOCITIES

ABSTRACT

The Hamiltonian H and Hamilton equations of motion are derived for the Fourier amplitudes $p_{\vec{k}}$ and $q_{\vec{k}}$ of the canonical conjugate fields $\vec{p}$ and $\vec{q}$ defined $\vec{p} = m\vec{v}$ and $\partial\vec{q}/\partial t = n\vec{v}$, where $\vec{v}(\vec{r},t)$ and $n(\vec{r},t)$ are the velocity and density fields of ideal, compressible gases in the state of fully developed turbulence. A Liouville equation is presented for the distribution function $f(\vec{p}_{\vec{k}},\vec{q}_{\vec{k}};\{\vec{k}\})$ in the multidimensional phase space formed by the scalar components of the set $\{\vec{k}\}$ of wave mode vectors $p_{\vec{k}}$ and $q_{\vec{k}}$. As an application, that stationary solution $f = f(H)$ of the Liouville equation is calculated, which maximizes the turbulence entropy. It is shown that the distribution of the velocity and density fluctuations of compressible gases is Gaussian in fully developed turbulence, in agreement with the experiments.

INTRODUCTION

The turbulence problem has aroused the interest of many celebrated researchers, such as Reynolds[1], von Kármán[2], Taylor[3], Heisenberg[4], Kolmogorov[5], Prandtl[6], and Frenkiel[7], to name only a few. In spite of all progress made to date[8-10], many experimental observations on the turbulence phenomenon are still unexplained from first principles. One of the most elementary experimental results is the Gaussian velocity distribution $\sim \exp(-\vec{v}^2/\bar{c}^2)$ of the turbulent velocity fluctuation $\vec{v}$ in compressible gases.[11] The Gaussian velocity distribution of fully developed turbulence is a fundamental fact as the Maxwellian distribution $\sim \exp(-\frac{1}{2}m\vec{v}^2/KT)$ of the molecular velocities $\vec{v}$ in statistical equilibrium gases.

The Maxwellian velocity distribution, which is the result of many molecular collisions, is independent of the molecular interactions, which have brought about the statistical equilibrium. The experiments indicate that in fully developed turbulence, the distribution function of the velocity fluctuations shows only in the far-out wings $|\vec{v}| \to \infty$ small deviations from the Gaussian distribution.[11] Apparently, the distribution of the velocity fluctions in stationary turbulence depends hardly on the viscous momentum transfer, which is negligible for the velocity fluctuations of large scale $\lambda \to \infty$ but significant for the velocity fluctuations of small scale $\lambda \to 0$, concerning the sustainment of the turbulent state since $-\nabla^2\vec{v}_{\vec{k}} \sim \vec{k}^2\vec{v}_{\vec{k}}$ $(\lambda = 2\pi/k)$.

In other words, the Gaussian distribution of the velocity fluctuations in fully developed turbulence is mainly sustained by nonlinear mode coupling of the turbulence elements of different wave vectors $\vec{k}$. For this reason, it should be possible to derive the turbulent velocity distribution from a statistical hydrodynamics of the turbulent fluid notions, without taking viscosity into account. By Fourier analyzing the canonical conjugate variables $\vec{p}(\vec{r},t)$ and $\vec{q}(\vec{r},t)$ of the turbulent gas, it is shown that the Fourier amplitudes $\vec{p}_{\vec{k}}(t)$ and $\vec{q}_{\vec{k}}(t)$ satisfy Hamilton equations of motion, i.e. the random wave modes $\vec{k}$ behave like quasi-particles. This makes it possible to derive a Liouville equation in the hyper-space formed by the scalar components of the set $\{\vec{p}_{\vec{k}}, \vec{q}_{\vec{k}}\}$. From this fundamental equation, the quasi-Gaussian distribution for the turbulent velocity fluctuations is derived as that stationary solution which maximizes the entropy of turbulence.

HAMILTON EQUATIONS

In the statistical turbulence theory to be presented, the effects of viscosity are disregarded so that the random velocity field $\vec{v}(\vec{r},t)$ of the gas is irrotational, $\nabla\times\vec{v}=\vec{0}$, in the absence of external forces which could artifically produce velocity curls. The condition $\nabla\times\vec{v}=\vec{0}$ defines longitudinal or so-called "acoustic" turbulence in compressible gases.[12] It represents a good approximation for the treatment of certain turbulence phenomenon, such as the present calculation of the turbulent velocity distribution.

The basis for the continuum-statistical considerations form the Hamilton equations of motion for the compressible, nonviscous gas. If $\vec{v}(\vec{r},t)$ and $n(\vec{r},t)$ designate the velocity and density field of the gas, respectively, the canonical conjugate fields $\vec{p}(\vec{r},t)$ and $\vec{q}(\vec{r},t)$ of the gas are defined by the proportionality (m=mass of gas molecule) and functional relations, respectively:

$$\vec{p} = m\vec{v} \quad , \qquad \partial\vec{q}/\partial t = n\vec{v} \qquad . \tag{1}$$

The Hamilton equations of motion and the Hamiltonian H of the gaseous system of volume $\Omega = \iiint d^3\vec{r} = L_x L_y L_z$ are given by:[13]

$$\frac{\partial\vec{q}}{\partial t} = \frac{\delta H}{\delta\vec{p}} \qquad , \tag{2}$$

$$\frac{\partial\vec{p}}{\partial t} = -\frac{\delta H}{\delta\vec{q}} \qquad , \tag{3}$$

where

$$H = \iiint_{\Omega} \mathcal{H} d^3\vec{r} \quad , \qquad \mathcal{H} = \frac{1}{2m}(\nu-\nabla\cdot\vec{q})\vec{p}^{\,2} + \frac{P_o}{\gamma-1}\left(\frac{\nu-\nabla\cdot\vec{q}}{n_o}\right)^{\gamma} \tag{4}$$

and

$$\frac{\delta H}{\delta\vec{p}} = \frac{\partial\mathcal{H}}{\partial\vec{p}} \quad , \qquad \frac{\delta H}{\delta\vec{q}} = \frac{\partial\mathcal{H}}{\partial\vec{q}} - \nabla\left(\frac{\partial\mathcal{H}}{\partial\nabla\cdot\vec{q}}\right) \tag{5}$$

defines the functional derivatives[14] [$\partial\mathcal{H}/\partial\vec{q} = \vec{0}$ since $\mathcal{H} = \mathcal{H}(\vec{p},\nabla\cdot\vec{q})$ by Eq.(4)]. In Eq.(4), γ_o ($\gamma_o = c_p/c_V$) is the adiabatic coefficient, P_o and n_o

[reference values of $P(\vec{r},t)$ and $n(\vec{r},t)$] and ν are constants obtained through integration of the polytropic energy conservation and continuity equations of the gas, respectively.

Since $n(\vec{r},t)$ and $\vec{v}(\vec{r},t)$ are random turbulent fields, $\vec{p}(\vec{r},t)$ and $\vec{q}(\vec{r},t)$ are represented as complex Fourier series,

$$\vec{p}(\vec{r},t) = \sum_{\{k\}} \vec{p}_{\vec{k}}(t) \exp(i\vec{k}\cdot\vec{r}), \qquad \vec{q}(\vec{r},t) = \sum_{\{k\}} \vec{q}_{\vec{k}}(t) \exp(i\vec{k}\cdot\vec{r}), \quad (6)$$

where

$$\vec{k} = 2\pi\left\{\frac{\nu_x}{L_x}, \frac{\nu_y}{L_y}, \frac{\nu_z}{L_z}\right\}, \qquad \nu_{x,y,z} = 0, \pm 1, \pm 2, \ldots, \pm \nu_\infty \quad (7)$$

are the discrete eigenvalues ($\Omega<\infty$) of the wave vector $\vec{k}$. Since the minimum wave length $\lambda = 2\pi/k_\infty$ is of the order of the mean free path, ℓ, one has $\nu_\infty \cong [L/\ell] >> 1$ for $\ell << L$.

Hamiltonian equations of motion similar to those for the canonical-conjugate fields $\vec{p}(\vec{r},t)$ and $\vec{q}(\vec{r},t)$ can be derived for their Fourier amplitudes $\vec{p}_{\vec{k}}(t)$ and $\vec{q}_{\vec{k}}(t)$, i.e., for the individual wave modes. By Eqs. (2)-(6),

$$\frac{\partial H}{\partial \vec{p}_{-\vec{k}}} = \iiint_\Omega \frac{\partial \vec{p}}{\partial \vec{p}_{-\vec{k}}} \cdot \frac{\partial H}{\partial \vec{p}} d^3\vec{r} = \iiint_\Omega e^{-i\vec{k}\cdot\vec{r}} \overleftrightarrow{\delta} \cdot \frac{\partial \vec{q}}{\partial t} d^3\vec{r} = \Omega d\vec{q}_{\vec{k}}/dt \quad (8)$$

and

$$\frac{\partial H}{\partial \vec{q}_{-\vec{k}}} = \iiint_\Omega \left\{ \frac{\partial \vec{q}}{\partial \vec{q}_{-\vec{k}}} \cdot \left[\frac{\partial H}{\partial \vec{q}} - \nabla\left(\frac{\partial H}{\partial \nabla\cdot\vec{q}}\right)\right] + \nabla \cdot \left(\frac{\partial \vec{q}}{\partial \vec{q}_{-\vec{k}}} \frac{\partial H}{\partial \nabla\cdot\vec{q}}\right)\right\} d^3\vec{r} = \iiint_\Omega e^{-i\vec{k}\cdot\vec{r}} \overleftrightarrow{\delta} \cdot \left(-\frac{\partial \vec{p}}{\partial t}\right) d^3\vec{r} =$$

$$= -\Omega d\vec{p}_{\vec{k}}/dt \quad , \quad (9)$$

since $\iiint \nabla\cdot\overleftrightarrow{A}\, d^3r = \vec{0}$ if $\overleftrightarrow{A}$ is expandable in orthogonal functions $e^{i\vec{k}\cdot\vec{r}}$ ($\overleftrightarrow{\delta}$ = unit tensor).

The canonical equations for the canonical-conjugate amplitudes $\vec{p}_{\vec{k}}(t)$ and $\vec{q}_{\vec{k}}(t)$ of the individual wave modes $\vec{k}$ are by Eqs. (8)-(9):

$$\frac{d\vec{p}_{\vec{k}}}{dt} = -\frac{1}{\Omega}\frac{\partial H}{\partial \vec{q}_{-\vec{k}}} \quad , \quad (10)$$

and

$$\frac{d\vec{q}_{\vec{k}}}{dt} = \frac{1}{\Omega}\frac{\partial H}{\partial \vec{p}_{-\vec{k}}} \quad . \quad (11)$$

Equations (10) and (11) represent a complete system of coupled nonlinear differential equations, which are antisymmetric in the $\vec{k}$ indices. These canonical equations determine the temporal development $p_{\vec{k}}(t)$ and $q_{\vec{k}}(t)$ of the wave modes $\vec{k}$ of spatial structure $s_{\vec{k}} = e^{i\vec{k}\cdot\vec{r}}$.

In Eqs. (10)-(11) the canonical equations of the wave mode dynamics are stated in complex form for the conjugate variables

$$\vec{p}_{\vec{k}} = \vec{P}_{\vec{k}} + i\vec{R}_{\vec{k}} \;, \qquad \vec{P}_{\vec{k}},\ \vec{R}_{\vec{k}} = \text{real} \;, \tag{12}$$

$$\vec{q}_{\vec{k}} = \vec{Q}_{\vec{k}} + i\vec{S}_{\vec{k}} \;, \qquad \vec{Q}_{\vec{k}},\ \vec{S}_{\vec{k}} = \text{real} \;, \tag{13}$$

so that

$$\vec{P}_{\vec{k}} = \frac{1}{2}(\vec{p}_{\vec{k}} + \vec{p}_{-\vec{k}}) \;, \qquad \vec{R}_{\vec{k}} = (1/2i)(\vec{p}_{\vec{k}} - \vec{p}_{-\vec{k}}) \;, \tag{14}$$

$$\vec{Q}_{\vec{k}} = \frac{1}{2}(\vec{q}_{\vec{k}} + \vec{q}_{-\vec{k}}) \;, \qquad \vec{S}_{\vec{k}} = (1/2i)(\vec{q}_{\vec{k}} - \vec{q}_{-\vec{k}}) \;, \tag{15}$$

since

$$\vec{p}^{\,*}_{\vec{k}} = \vec{p}_{-\vec{k}} \;, \qquad \vec{q}^{\,*}_{\vec{k}} = \vec{q}_{-\vec{k}} \tag{16}$$

for real vector fields $\vec{p}(\vec{r},t)$ and $\vec{q}(\vec{r},t)$, Eq.(6). Substitution of Eqs.(12)-(13) into Eqs. (10)-(11) yields the Hamilton equations for the real and imaginary parts of the canonical conjugate mode amplitudes:

$$\frac{d\vec{P}_{\vec{k}}}{dt} = -\frac{1}{\Omega}\frac{\partial H}{\partial \vec{Q}_{\vec{k}}} \;, \qquad \frac{d\vec{R}_{\vec{k}}}{dt} = -\frac{1}{\Omega}\frac{\partial H}{\partial \vec{S}_{\vec{k}}} \;, \tag{17}$$

$$\frac{d\vec{Q}_{\vec{k}}}{dt} = \frac{1}{\Omega}\frac{\partial H}{\partial \vec{P}_{\vec{k}}} \;, \qquad \frac{d\vec{S}_{\vec{k}}}{dt} = \frac{1}{\Omega}\frac{\partial H}{\partial \vec{R}_{\vec{k}}} \;. \tag{18}$$

LIOUVILLE EQUATION

The existence of the canonical equations (10) and (11), which are analogous to the canonical equations of a many-body system, indicates a quasiparticle behavior of the individual wave modes. For statistical purposes, let a large number (ensemble) of similar, turbulent gaseous systems be introduced (each containing a set $\{\vec{k}\}$ of wave modes or quasiparticles $\vec{k}$) in the multidemensional phase space formed by the real and imaginary parts of the Fourier vector components $p_{\vec{k},i}$ and $q_{\vec{k},i}$ $(i=1,2,3)$ of the entire set $\{\vec{k}\}$ defined in Eq.(7). The number of gas systems, which have their phases in the volume element $\Pi_{\{\vec{k}\}}(d^6\vec{p}_{\vec{k}}d^6\vec{q}_{\vec{k}})$ at the point $(\vec{p}_{\vec{k}}, \vec{q}_{\vec{k}};\{\vec{k}\})$ of the phase space, is given in terms of the density in phase space, $f=f(\vec{p}_{\vec{k}}, \vec{q}_{\vec{k}}, t; \{\vec{k}\})$, by

$$dW = f(\vec{p}_{\vec{k}}, \vec{q}_{\vec{k}}, t; \{\vec{k}\}) \prod_{\{\vec{k}\}} (d^6\vec{p}_{\vec{k}} d^6\vec{q}_{\vec{k}}), \tag{19}$$

where

$$d^6\vec{p}_{\vec{k}} = d^3\vec{P}_{\vec{k}} d^3\vec{R}_{\vec{k}}, \quad d^6\vec{q}_{\vec{k}} = d^3\vec{Q}_{\vec{k}} d^3\vec{S}_{\vec{k}}, \tag{20}$$

in accordance with Eqs. (12) and (13). The phase-space density satisfies the continuity equation in phase space, since the number of plasma systems in the ensemble is conserved.

$$\frac{\partial f}{\partial t} + \nabla_{(\vec{k})} \cdot (\vec{V}_{(\vec{k})} f) = 0 , \tag{21}$$

where

$$\vec{V}_{\{\vec{k}\}} = \sum_{\{\vec{k}\}} \left(\frac{d\vec{p}_{\vec{k}}}{dt} , \frac{d\vec{q}_{\vec{k}}}{dt} \right), \tag{22}$$

$$\nabla_{\{\vec{k}\}} = \sum_{\{\vec{k}\}} \left(\frac{\partial}{\partial \vec{p}_{\vec{k}}} , \frac{\partial}{\partial \vec{q}_{\vec{k}}} \right), \tag{23}$$

The canonical equations (10) and (11), which determine the motion of any point in phase space, indicate that the phase-space fluid is incompressible,

$$\nabla_{\{\vec{k}\}} \cdot \vec{V}_{\{\vec{k}\}} = \frac{1}{\Omega} \sum_{\{\vec{k}\}} \left[\frac{\partial}{\partial \vec{p}_{\vec{k}}} \cdot \left(- \frac{\partial H}{\partial \vec{q}_{-\vec{k}}} \right) + \frac{\partial}{\partial \vec{q}_{\vec{k}}} \cdot \left(\frac{\partial H}{\partial \vec{p}_{-\vec{k}}} \right) \right] = 0 . \qquad (24)$$

Because of this property, the continuity equation (21) can be brought into the form of the so-called Liouville equation:

$$\frac{\partial f}{\partial t} = - \frac{1}{\Omega} \sum_{\{\vec{k}\}} \left(\frac{\partial f}{\partial \vec{q}_{\vec{k}}} \cdot \frac{\partial H}{\partial \vec{p}_{-\vec{k}}} - \frac{\partial H}{\partial \vec{q}_{-\vec{k}}} \cdot \frac{\partial f}{\partial \vec{p}_{\vec{k}}} \right) . \qquad (25)$$

This fundamental equation describes the temporal development of the distribution in phase space. It extends statistical mechanics to gas continua in random motion.

The Hamiltonian H in Eq. (4) is an invariant for a nondissipative gas. The steady-state solutions of Eq. (25) are functions of H:

$$f = f(H) \qquad , \qquad (26)$$

since

$$\frac{\partial f}{\partial t} = - \frac{1}{\Omega} \sum_{\{\vec{k}\}} \left(\frac{\partial H}{\partial \vec{q}_{\vec{k}}} \cdot \frac{\partial H}{\partial \vec{p}_{-\vec{k}}} - \frac{\partial H}{\partial \vec{q}_{-\vec{k}}} \cdot \frac{\partial H}{\partial \vec{p}_{\vec{k}}} \right) \frac{\partial f}{\partial H} = 0, \quad \text{for } f=f(H) \quad . \qquad (27)$$

In applications, the compact complex form (25) of the Liouville equation is mathematically preferable. In terms of real vector components, Eqs. (22)-(23) become

$$\vec{V}_{\{\vec{k}\}} = \sum_{\{\vec{k}\}} \left(\frac{d\vec{P}_{\vec{k}}}{dt} , \frac{d\vec{R}_{\vec{k}}}{dt} , \frac{d\vec{Q}_{\vec{k}}}{dt} , \frac{d\vec{S}_{\vec{k}}}{dt} \right) , \qquad (28)$$

$$\nabla_{\{\vec{k}\}} = \sum_{\{\vec{k}\}} \left(\frac{\partial}{\partial \vec{P}_{\vec{k}}} , \frac{\partial}{\partial \vec{R}_{\vec{k}}} , \frac{\partial}{\partial \vec{Q}_{\vec{k}}} , \frac{\partial}{\partial \vec{S}_{\vec{k}}} \right) \quad . \qquad (29)$$

Since the phase space fluid is incompressible, $\nabla_{\{\vec{k}\}} \cdot \vec{V}_{\{\vec{k}\}} = 0$ by Eq.(24), substitution of Eqs.(28)-(29) into the continuity equation (21) results in the Liouville equation in real notation:

$$\frac{\partial f}{\partial t} = - \frac{1}{\Omega} \sum_{\{\vec{k}\}} \left[\frac{\partial f}{\partial \vec{Q}_{\vec{k}}} \cdot \frac{\partial H}{\partial \vec{P}_{\vec{k}}} + \frac{\partial f}{\partial \vec{S}_{\vec{k}}} \cdot \frac{\partial H}{\partial \vec{R}_{\vec{k}}} - \frac{\partial H}{\partial \vec{Q}_{\vec{k}}} \cdot \frac{\partial f}{\partial \vec{P}_{\vec{k}}} - \frac{\partial H}{\partial \vec{S}_{\vec{k}}} \cdot \frac{\partial f}{\partial \vec{R}_{\vec{k}}} \right] . \qquad (30)$$

The solution of the Louville equation permits the determination of the macroscopic properties of turbulent gases as ensemble averages. The statistical considerations are based on the quasi-particle character of the individual wave modes.

TURBULENCE ENTROPY

The state of stationary, fully developed turbulence is one of maximum probability W. Since the entropy of a turbulent system is a function $S = \ln W$ of its probability W, the entropy of turbulence must assume an extremum in stationary state.

The entropy associated with the turbulent modes of the gas is defined as the phase space integral over the distribution function $f = f(\vec{p}_{\vec{k}}, \vec{q}_{\vec{k}}; \{\vec{k}\})$:

$$S = -\kappa \int \ldots \int f \ln f \prod_{\{\vec{k}\}} (d^6 \vec{p}_{\vec{k}} d^6 \vec{q}_{\vec{k}}) \tag{31}$$

where κ is a dimensional constant. This definition of turbulence entropy is in complete analogy of the definition of turbulence for arbitrary nonequilibrium systems.[15)]

The idealized description of the gas as a continuum without collisional dissipation is reflected in the time - independence of the turbulence entropy. By Eqs. (3.), (21), and (24).

$$\frac{dS}{dt} = -k \int \ldots \int (1 + \ln f) \left(\frac{\partial}{\partial t} + \vec{V}_{\{\vec{k}\}} \cdot \nabla_{\{\vec{k}\}} \right) f \prod_{\{\vec{k}\}} (d^6 \vec{p}_{\vec{k}} d^6 \vec{q}_{\vec{k}}) = 0. \quad (32).$$

VELOCITY DISTRIBUTION

As an application of the canonical theory, the statistical distribution of the random velocities in turbulent, compressible gases is calculated. Let the turbulent gas be, on the average, homogeneous ($\bar{n} = n_0$) and at rest ($\vec{\bar{v}} = \vec{0}$) so that for the spatial mean values

$$\vec{\bar{p}} = \vec{0} \quad , \qquad \vec{\bar{q}} \neq \vec{0} \ . \tag{33}$$

The gas is assumed to be in contact with an appropriate other system which sustains stationary, fully developed turbulence. The canonical conjugate fluctuation fields can then be expanded in the stationary Fourier series:[10)]

$$\vec{p}(\vec{r}) = \sum_{\{\vec{k}\}} \vec{p}_{\vec{k}} \exp[i\vec{k}\cdot\vec{r}], \qquad \vec{q}(\vec{r}) = \sum_{\{\vec{k}\}} \vec{q}_{\vec{k}} \exp[i\vec{k}\cdot\vec{r}], \tag{34}$$

where

$$\vec{p}_{\vec{k}} = \frac{1}{\Omega}\iiint_{\Omega} \vec{p}(\vec{r})\exp[-i\vec{k}\cdot\vec{r}]d^3\vec{r} \ , \tag{35}$$

$$\vec{q}_{\vec{k}} = \frac{1}{\Omega}\iiint_{\Omega} \vec{q}(\vec{r})\exp[-i\vec{k}\cdot\vec{r}]d^3\vec{r} \tag{36}$$

are the complex, time-independent Fourier amplitudes of the real fields $\vec{p}(\vec{r})$ and $\vec{q}(\vec{r})$:

$$\vec{p}^{\,*}_{\vec{k}} = \vec{p}_{-\vec{k}}, \qquad \vec{q}^{\,*}_{\vec{k}} = \vec{q}_{-\vec{k}} \quad . \tag{37}$$

The orthogonality relations for elementary vector fields of the form given in the above Fourier expansions are

$$\iiint (\vec{p})_{\vec{k}}\cdot(\vec{q})^*_{\vec{k}'} d^3\vec{r} = \iiint \vec{p}_{\vec{k}}\cdot\vec{q}^{\,*}_{\vec{k}'}\exp[i(\vec{k} - \vec{k}')\cdot\vec{r}]d^3\vec{r} = \Omega\ \vec{p}_{\vec{k}}\ \vec{q}^{\,*}_{\vec{k}'}\delta_{\vec{k}\vec{k}'} \ . \tag{38}$$

By Eq.(27), the phase space density of a conservative gas (H=E) in a stationary state is an arbitrary function of the Hamiltonian

$$\rho(\vec{p}_{\vec{k}}, \vec{q}_{\vec{k}}, \{\vec{k}\}) = f[H(\vec{p}_{\vec{k}}, \vec{q}_{\vec{k}}; \{\vec{k}\})] \quad . \tag{39}$$

[illegible] stationary state, the entropy of turbulence assumes an extremum (state [illegible] stability). By Eq. (31)

$$S(\vec{p}_{\vec{k}},\vec{q}_{\vec{k}};\{\vec{k}\}) = -\kappa\int\ldots\int f(\vec{p}_{\vec{k}},\vec{q}_{\vec{k}};\ \{\vec{k}\}\ )\ \ell n\ f(\vec{p}_{\vec{k}},\vec{q}_{\vec{k}};\ \{\vec{k}\}\ )\ \prod_{\{\vec{k}\}}(d^6\vec{p}_{\vec{k}}d^6\vec{q}_{\vec{k}})\ . \tag{40}$$

Accordingly, the phase-space distribution $f(H)$ of stationary, fully developed turbulence, is determined by the variational equation

$$\delta S = -\kappa\int\ldots\int(1 + \ell n\ f)\ \delta f\ \prod_{\{\vec{k}\}}(d^6\vec{p}_{\vec{k}}d^6\vec{q}_{\vec{k}}) = 0\quad , \tag{41}$$

where

$$\int\ldots\int\delta f\prod_{\{\vec{k}\}}(d^6\vec{p}_{\vec{k}}d^6\vec{q}_{\vec{k}}) = 0\quad , \tag{42}$$

$$\int\ldots\int H\ \delta f\ \prod_{\{\vec{k}\}}(d^6\vec{p}_{\vec{k}}d^6\vec{q}_{\vec{k}}) = 0\ , \tag{43}$$

since the number of gas systems and the total energy are invariant. Upon multiplication of Eq. (42) by $(\alpha-1)$ and Eq. (43) by β, the method of undetermined Lagrangian multipliers gives

$$\int\ldots\int(\ell n\ f + \alpha + \beta H)\ \delta f\ \prod_{\{\vec{k}\}}(d^6\vec{p}_{\vec{k}}d^6\vec{q}_{\vec{k}}) = 0 \tag{44}$$

i.e.

$$\ell n\ f + \alpha + \beta H = 0 \tag{45}$$

since δf is arbitrary. From Eq. (45) the distribution function $f(H)$ follows after normalization,

$$\alpha = \ell n\left[\int\ldots\int\exp\ [-\beta H]\cdot\prod_{\{\vec{k}\}}(d^6\vec{p}_{\vec{k}},d^6\vec{q}_{\vec{k}})\right]\quad , \tag{46}$$

in the form

$$f(H) = \frac{\exp[-\beta H(\vec{p}_{\vec{k}},\vec{q}_{\vec{k}};\{\vec{k}\})]}{\int\ldots\int\exp[-\beta H(\vec{p}_{\vec{k}},\vec{q}_{\vec{k}};\{\vec{k}\})]\ \prod_{\{\vec{k}\}}\ (d^6\vec{p}_{\vec{k}}d^6\vec{q}_{\vec{k}})}. \tag{47}$$

The Hamiltonian in Eq. (4) becomes after substitution of the expansions in Eqs. (35)-(36)

$$H = \iiint\left[\frac{1}{2m}\left(n_o - i\sum_{\{\vec{k}\}}\vec{k}\cdot\vec{q}_{\vec{k}}\exp[i\vec{k}\cdot\vec{r}]\right)\left(\sum_{\{\vec{k}\}}\vec{p}_{\vec{k}}\exp[i\vec{k}\cdot\vec{r}]\right)^2 + \right. \tag{48}$$

$$\left. + \frac{P_o}{\gamma-1}\left(1 - \frac{i}{n_o}\sum_{\{\vec{k}\}}\vec{k}\cdot\vec{q}_{\vec{k}}\exp[i\vec{k}\cdot\vec{r}]\right)^{\gamma}\ \right]d^3\vec{r} \tag{49}$$

since $\nu = n_o$. An evident bionomial expansion and subsequent spatial integration gives, under consideration of Eq. (38),

$$H = \Omega\left[\frac{P_o}{\gamma-1} + \frac{n_o}{2m}\sum_{\{\vec k\}}\vec p_{\vec k}\cdot\vec p_{\vec k}^{\,*} + \frac{1}{2}\gamma P_o n_o^{-2}\sum_{\{\vec k\}}(i\vec k\cdot\vec q_{\vec k})(i\vec k\cdot\vec q_{\vec k}^{\,*})\right] + h(\vec p_{\vec k},\vec q_{\vec k};\{\vec k\}) \qquad (50)$$

where

$$h(\vec p_{\vec k},\vec q_{\vec k};\{\vec k\}) = \Omega\Bigg\{-\frac{i}{2m}\sum_{\{\vec k,\vec k'\}}(\vec k\cdot\vec q_{\vec k})\vec p_{\vec k'}\cdot\vec p_{-(\vec k+\vec k')} -$$

$$-\frac{P_o}{\gamma-1}\sum_{l=3}^{\infty}\binom{\gamma}{l}\frac{(-i)^l}{n_o^l}\sum_{\{\vec k^{(1)}\ldots\vec k^{(l-1)}\}}([\vec k^{(1)}\cdot\vec q_{\vec k}^{(1)}]\ldots[\vec k^{(l-1)}\cdot\vec q_{\vec k}^{(l-1)}])\cdot$$

$$\cdot[\vec k^{(1)} + \ldots\vec k^{(l-1)}]\cdot\vec q_{-(\vec k^{(1)}+\ldots+\vec k^{(l-1)})}\Bigg\} \qquad (51)$$

contains the terms of third and higher order in the Fourier amplitudes. The various sums in Eq. (50) have the physical meaning of spatial averages (Parseval's theorem)

$$\sum_{\{\vec k\}}\vec p_{\vec k}\cdot\vec p_{\vec k}^{\,*} = \overline{\vec p^{\,2}} = m^2\,\overline{\vec v^{\,2}} \quad , \qquad (52)$$

$$\sum_{\{\vec k\}}(i\vec k\cdot\vec q_{\vec k})(i\vec k\cdot\vec q_{\vec k}^{\,*}) = \sum_{\{\vec k\}}\tilde n_{\vec k}\,\tilde n_{\vec k}^{\,*} = \overline{\tilde n^2} \quad , \qquad (53)$$

where $\tilde{\vec v} = \vec v$ ($\vec v_o = \vec 0$) and $\tilde n = n - n_o$ are the turbulent velocity and density fluctuations of the compressible gas.

Combining of Eqs. (47) and (50) yields the stationary distribution function in the form:

$$f(H) = \frac{\exp\left\{-\beta\Omega\left[\frac{n_o}{2m}\sum_{\{\vec k\}}\vec p_{\vec k}\cdot\vec p_{\vec k}^{\,*} + \frac{1}{2}\gamma P_o n_o^{-2}\sum_{\{\vec k\}}(i\vec k\cdot\vec q_{\vec k})(i\vec k\cdot\vec q_{\vec k}^{\,*})\right] - \beta h(\vec p_{\vec k},\vec q_{\vec k};\{\vec k\})\right\}}{[\exp(\beta\Omega P_o(\gamma-1)^{-1})]\int\ldots\int\exp[-\beta H(\vec p_{\vec k},\vec q_{\vec k};\{\vec k\})]\prod_{\{\vec k\}}d^6\vec p_{\vec k}\,d^6\vec q_{\vec k}} \; . \qquad (54)$$

By introducing the spatial averages defined in Eqs. (52)-(53) a physically more illustrative representation of the stationary turbulence distribution is obtained:

$$f(H) = A\exp\left\{-\beta\Omega\left[\frac{1}{2}n_o m\,\overline{\vec v^{\,2}} + \frac{1}{2}\gamma P_o\overline{(\tilde n/n_o)^2}\right] - \beta h(\vec p_{\vec k},\vec q_{\vec k};\{\vec k\})\right\} \qquad (55)$$

where

$$A^{-1} = [\exp(\beta\Omega P_o(\gamma-1)^{-1})]\int\ldots\int \exp[-\beta H(\vec{p}_{\vec{k}},\vec{q}_{\vec{k}};\{\vec{k}\})] \prod_{\{\vec{k}\}} d^6\vec{p}_{\vec{k}}\, d^6\vec{q}_{\vec{k}} \qquad (56)$$

The parameter β of the distribution, which has the dimension of a reciprocal energy, is given in terms of the total turbulent energy $\tilde{E}$ contained in the fluctuations through the relation

$$\tilde{E} = \int\ldots\int H(\vec{p}_{\vec{k}},\vec{q}_{\vec{k}};\{\vec{k}\})\, f(\beta H(\vec{p}_{\vec{k}},\vec{q}_{\vec{k}};\{\vec{k}\})) \prod_{\{\vec{k}\}} d^6\vec{p}_{\vec{k}}\, d^6\vec{q}_{\vec{k}} - \frac{P_o\Omega}{\gamma-1}\,, \qquad (57)$$

in view of the normalization defined in Eq. (56). $P_o\Omega/(\gamma-1)$ represents the nonturbulent or thermal energy of the gaseous system of volume Ω.

Eq.(55) indicates that the distribution function $f(H)$ for the turbulent fluctuations has the following properties:

i) The distribution of the velocity fluctuations is Gaussian,

$$f(\vec{v}^2) \sim e^{-\frac{1}{2}n_o m\beta\Omega\vec{v}^2}\,. \qquad (58)$$

ii) The distribution of the density fluctuations is Gaussian,

$$f(\tilde{n}^2) \sim e^{-\frac{1}{2}\gamma P_o n_o^{-2}\beta\Omega\tilde{n}^2}\,. \qquad (59)$$

The higher order terms in the distribution function $f(H)$ of Eq.(55) due to $h(\vec{p}_{\vec{k}},\vec{q}_{\vec{k}};\{\vec{k}\})$, Eq.(51), are difficult to observe in experiments for the following reasons. They represent cross-effects of density and velocity fluctuations, i.e. they are unobservable if the experimental detector record only velocity or density fluctuations. The term $h(\vec{p}_{\vec{k}},\vec{q}_{\vec{k}};\{\vec{k}\})$ is small of third and higher order in the fluctuation amplitudes and is, in general, significantly smaller than the leading second order terms in the expansion of $H(\vec{p}_{\vec{k}},\vec{q}_{\vec{k}};\{\vec{k}\})$.

Normal distributions of the forms given in Eqs. (58)-(59) have been observed in fully developed turbulence of compressible gases.[11] Experimentally the distribution of turbulent velocities is obtained by plotting the frequency of the occurence (over a larger time period) of a velocity signal with amplitudes between $|\vec{v}|$ and $|\vec{v}| + |\Delta\vec{v}|$ vs. $|\vec{v}|$, i.e. by a time-averaging process.

In the distribution function of Eq. (55), the quantities $\overline{\vec{v}^2}$ and $\overline{\tilde{n}^2}$ represent spatial averages (Eqs. (52) and (53). This formal difference is, of course, no contradiction between experiment and theory, since the time average is equal to the spatial (ensemble) average in fully developed, homogeneous turbulence. In Fig. 1, the experimental (o) and theoretical (-) distributions of the turbulent velocities are compared.[11] It is seen that the theoretical Gaussian distribution of the turbulent velocities is in excellent agreement with the experimental data.

The theory presented is highly idealized since viscous and thermal dissipation in the turbulent gas are not taken, explicitly, into account. It represents, however, a first attempt at extending statistical mechanics to random continuous media, such as turbulent gases. Extensions of the theory to include viscous and thermal dissipation represent mainly mathematical problems.

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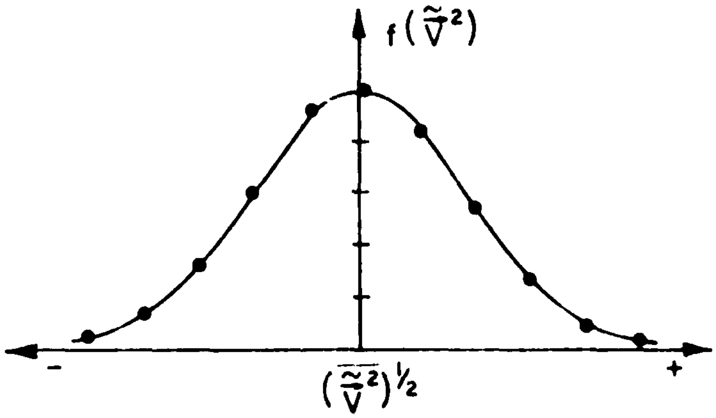

FIG. 1: Experimental[11] (•) and theoretical (-) distribution of velocity fluctuations.

VII. APPENDIX: STRESS RELAXATION WAVES IN FLUIDS

ABSTRACT

The Navier-Stokes equations for incompressible and compressible fluids are generalized by inclusion of viscous stress relaxation, as required by kinetic theory. Two initial-boundary-value problems of the nonlinear, generalized Navier-Stokes equations are solved analytically, which describe the propagation of transverse or shear waves due to temporal and spatial velocity pulses $\vec{v}(0,t)$ and $\vec{v}(x,0)$, respectively. It is shown that transverse perturbations propagate in form of a discontinuous wave with a finite wave speed due to viscous stress relaxation, whereas the conventional Navier-Stokes equations result in non-physical solutions suggesting a diffusion process covering the entire fluid with infinite speed.

INTRODUCTION

The nonlinear incompressible and compressible Navier-Stokes equations represent (quasi) parabolic and hyperbolic partial differential equations, respectively. The former propagate signals with infinite speed and the latter propagate certain signals with finite speed in fluids. In an infinite, homogeneous fluid, consider a small (linear) velocity perturbation, which is representable as the Fourier integral,

$$\tilde{\vec{v}}(r,t) = \int_{-\infty}^{+\infty} \hat{\vec{v}}(\vec{k})\, e^{i\omega t - i\vec{k}\cdot\vec{r}}\, d\omega \quad ,$$

over elementary waves of wave length $\lambda = 2\pi/k$ and frequency $\omega(k)$. If the fluid is compressible so that it sustains both pressure $(\tilde{p})$ and density $(\tilde{\rho})$ perturbations, $\partial\tilde{p}/\partial t = c_s^2\, \partial\tilde{\rho}/\partial t$, the perturbation can propagate, e.g., in form of longitudinal sound waves with finite speed $c_s = (\gamma p_o/\rho_o)^{\frac{1}{2}}$ and dispersion law

$$\omega^2 = c_s^2 k^2 + i(5\mu/3\rho_o)\omega k^2 \quad .$$

In a fluid with a viscosity μ, a perturbation may also propagate in form of a transverse or shear wave. If one applies the curl operation to the incompressible or compressible Navier-Stokes equation, a dispersion law is found for the transverse perturbations which does not represent a wave phenomenon but an aperiodic damping process with

$$i\omega = -(\mu/\rho_o)k^2 \quad .$$

As is known, the acoustic dispersion law is derived from a hyperbolic wave equation, whereas the damping constant for the transverse modes follows from the parabolic vorticity equation (which is the same for incompressible and compressible fluids). From experiments, however, it is established that transverse perturbations ($\nabla \times \tilde{\vec{v}}_k = i\vec{k} \times \tilde{\vec{v}}_k \neq \vec{0}$) propagate as (hyperbolic) shear waves with finite speed.

It will be demonstrated herein that the (incompressible or compressible) Navier-Stokes equations do not provide a correct description of shear waves. The discrepancy between the Navier-Stokes equations and the experiments on shear waves is resolved by introducing viscous stress relaxation, which leads to a hyperbolic transport equation for shear waves in incompressible or compressible fluids. For this reason, the transverse or shear waves represent "stress relaxation waves".

As an illustration, two hyperbolic initial-boundary-value problems for shear waves with stress relaxation are solved. The solutions of the generalized Navier-Stokes equations with stress relaxation represent transverse waves which are discontinuous at the wave front and have a finite wave speed, $c = (\mu/\rho_o\tau)^{\frac{1}{2}} < \infty$ (τ is the stress relaxation time). The first treats the propagation of a shear wave into a semi-infinite fluid space, $x \geq 0$, produced by a temporal velocity impulse at the boundary $x = 0$ (accelerated wall). The second is concerned with the propagation of a shear wave into an infinite fluid space, $-\infty \leq x \leq +\infty$, caused by a spatial velocity pulse in the plane $x = 0$ at time $t = 0$. Both solutions are valid for nonlinear shear waves of arbitrary intensity.

PHYSICAL PRINCIPLES

In conventional fluid mechanics,[1] it is assumed that inhomogeneities $\nabla_i v_j$ in the velocity components v_j produce instantaneously viscous stresses Π_{ij}. Mathematically, this is expressed through the phenomenological "flux" ~ "force" relation

$$\Pi_{ij} = -\mu(\nabla_i v_j + \nabla_j v_i - \frac{2}{3}\nabla_k v_k \delta_{ij})$$

where μ is the viscosity and $\overleftrightarrow{\delta}$ is the unit tensor. In a real continuum, however, velocity inhomogeneities do not switch on viscous stresses instantaneously but in accordance with a relaxation process of characteristic time τ. By means of the kinetic theory of gases[2] and liquids,[3] one can show that the transport equation for the viscous stresses has the form of a temporal $(\partial/\partial t)$ and spatial $(\vec{v}\cdot\nabla)$ relaxation equation,

$$\frac{\partial}{\partial t}\Pi_{ij} + v_k \nabla_k \Pi_{ij} = -\tau^{-1}\Pi_{ij} - \mu\tau^{-1}(\nabla_i v_j + \nabla_j v_i - \frac{2}{3}\nabla_k v_k \delta_{ij}).$$

This equation is approximate insofar as the the coupling of heat flows q_i and stresses Π_{ij} and higher order terms in the derivatives of v_i are neglected.[2,3] It has temporal and spatial derivatives as required for a $\vec{r}$-t dependent field equation and is Galilei covariant. If relaxation effects are disregarded, it reduces to the static stress equation.

Thus, consideration of viscous stress relaxation leads to a reformulation of the conventional Navier-Stokes theory of incompressible and compressible fluids. In place of the Navier-Stokes equations, we have the hydrodynamic equations with viscous stress relaxation:

$$\rho(\frac{\partial \vec{v}}{\partial t} + \vec{v}.\nabla\vec{v}) = -\nabla p - \nabla\cdot\overleftrightarrow{\Pi} \quad , \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \vec{v}.\nabla\rho = -\rho\nabla\cdot\vec{v} \quad , \tag{2}$$

$$\frac{\partial \overleftrightarrow{\Pi}}{\partial t} + \vec{v}\cdot\nabla\overleftrightarrow{\Pi} + \frac{\overleftrightarrow{\Pi}}{\tau} = -\frac{\mu}{\tau}(\nabla\vec{v} + \overline{\nabla\vec{v}} - \frac{2}{3}\nabla\cdot\vec{v}\,\overleftrightarrow{\delta}). \tag{3}$$

Eqs. (1) - (3) hold for incompressible ($\nabla\cdot\vec{v} = 0$) and compressible ($\nabla\cdot\vec{v} \neq 0$) fl . For nonisothermal systems, the transport equations for thermal energy and heat flux have to be added to Eqs. (1) - (3).[4,5]

If μ and τ can be treated as $\vec{r}$-independent, it is mathematically more convenient to use instead of the tensor equation (3) the vector equation,

$$\frac{\partial}{\partial t}\nabla\cdot\overleftrightarrow{\Pi} + \nabla\cdot(\vec{v}\cdot\nabla\overleftrightarrow{\Pi}) + \tau^{-1}\nabla\cdot\overleftrightarrow{\Pi} = -\mu\tau^{-1}(\nabla^2\vec{v} + \frac{1}{3}\nabla\nabla\cdot\vec{v}), \tag{4}$$

since Eq. (1) contains the force density $\nabla\cdot\overleftrightarrow{\Pi}$. If temporal and spatial relaxation of the viscous stresses is disregarded, Eqs. (1) and (4) combine to the classical Navier-Stokes equation, $\rho(\partial\vec{v}/\partial t + \vec{v}\cdot\nabla\vec{v}) = -\nabla p + \mu\nabla^2\vec{v} + (\mu/3)\nabla\nabla\cdot\vec{v}$.

Equations (1) - (3) represent a hyperbolic system both in the compressible and incompressible cases. On the other hand, the conventional incompressible Navier-Stokes equations are parabolic. The corresponding field equations for incompressible fluids are obtained by setting $\nabla\cdot\vec{v} \equiv 0$ in Eqs. (2), (3), and (4).

INITIAL-BOUNDARY-VALUE PROBLEM FOR $\vec{v}(0,t)$-PULSE

A simple method for the generation of transverse waves in a viscous fluid consists in setting the plane x = 0 bounding a semi-infinite fluid ($x \geq 0$, $|y| \leq \infty$, $|z| \leq \infty$) into sudden motion $\vec{v}_w = \bar{v}\, H(t)\vec{e}_y$, where H(t) is the Heaviside step function. The resulting viscous interaction between the fluid and the accelerated wall produces a curl $\vec{n} \times [\vec{v}] = \vec{e}_z\, v(x = 0,t)$ at the fluid surface which propagates in form of a transverse wave through the fluid in the x-direction. In this dynamic process, the fluid velocity is of the form $\vec{v} = \{0,\ v(x,t),\ 0\}$ so that $\nabla\cdot\vec{v} = \partial v/\partial y = 0$ and $\vec{v}\cdot\nabla\vec{v} = \vec{0}$, i.e., the fluid motion behaves incompressible (even if the fluid is compressible) and linear. Furthermore, $\vec{v}\cdot\nabla\overleftrightarrow{\Pi} = v\ \partial\overleftrightarrow{\Pi}/\partial y = 0$ since $\overleftrightarrow{\Pi}$ has only a single component $\Pi_{xy} = \Pi(x,t)$ by Eq. (3), and $\nabla p = \vec{0}$ by Eq. (1).

Thus, Eqs. (1) - (3) lead to the following initial-boundary-value problem for the transverse velocity wave $\vec{v}(x,t)$ in the y-direction propagating in the x-direction, as a result of the sudden wall motion in the plane x = 0:

$$\rho_o \frac{\partial v}{\partial t} = -\frac{\partial \Pi}{\partial x} \quad , \qquad (5)$$

$$\frac{\partial \Pi}{\partial t} + \frac{\Pi}{\tau} = -\frac{\mu}{\tau}\frac{\partial v}{\partial x} \quad , \qquad (6)$$

$$v(x = 0,t) = \bar{v}\ H(t), \quad t \geq 0 \quad , \qquad (7)$$

$$v(x,t = 0) = 0, \ x > 0 \quad , \qquad (8)$$

$$\partial v(x,t = 0)/\partial t = 0, \ x > 0 \quad , \qquad (9)$$

where $H(t) = 0,\ t \leq -0$, and $H(t) = 1,\ t \geq +0$.

Equations (5) - (6) represent a hyperbolic system, from which one obtains by elimination wave equations for the stress component $\Pi_{xy} \equiv \Pi(x,t)$ and the velocity field v(x,t):

$$\frac{\partial^2 \Pi}{\partial t^2} + \frac{1}{\tau}\frac{\partial \Pi}{\partial t} = c^2\frac{\partial^2 \Pi}{\partial x^2} \qquad (10)$$

and

$$\frac{\partial^2 v}{\partial t^2} + \frac{1}{\tau}\frac{\partial v}{\partial t} = c^2 \frac{\partial^2 v}{\partial x^2} \tag{11}$$

where

$$c = (\mu/\rho_o \tau)^{\frac{1}{2}} \tag{12}$$

is the (maximum) speed of the stress relaxation wave. Both $H(x,t)$ and $v(x,t)$ satisfy similar (hyperbolic) wave equations with the same wave speed c. In the limit, $\tau \to 0$ and $c \to \infty$, with $c^2\tau \to \mu/\rho_o$, Eqs. (10) and (11) reduce to parabolic equations, according to which boundary values of $H(x,t)$ and $v(x,t)$ would diffuse with infinite speed into the fluid (conventional Navier-Stokes theory). Accordingly, only for $\tau > 0$ and $c < \infty$, transverse or shear waves exist in the fluid which represent, therefore, stress relaxation waves.

According to Eq. (11) and Eqs. (7) - (9), the velocity field $v(x,t) = \hat{v}\, u(\xi, t)$ of the stress relaxation wave under consideration is described by the dimensionless initial-boundary-value problem:

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \xi^2} \quad , \tag{13}$$

$$u(\xi = 0, t) = H(t), \quad t \geq 0, \quad , \tag{14}$$

$$u(\xi, t = 0) = 0, \quad \xi > 0 \quad , \tag{15}$$

$$\partial u(\xi, t = 0)/\partial t = 0, \quad \xi > 0 \quad , \tag{16}$$

where

$$u(\xi, t) = v(x,t)/\hat{v}, \quad \xi = x/c\tau, \quad t = t/\tau \quad . \tag{17}$$

Equation (13) - (16) are solved by means of the Laplace transform technique[6] which gives

$$\bar{u}(\xi, s) = L[u(\xi, t)] = \int_0^\infty e^{-st}\, u(\xi, t)\, dt \quad , \tag{18}$$

$$\bar{u}(0,s) = L[u(0,t)] = \int_0^\infty e^{-st}\, H(t)\, dt = s^{-1} .$$

Since the initial conditions (15) - (16) vanish, Eqs. (13) - (14) yield for the transformed velocity $\bar{u}(\xi,s)$ the ordinary boundary-value problem,

$$\frac{d^2\bar{u}}{d\xi^2} - (s^2 + s)\bar{u} = 0 \quad , \tag{20}$$

$$\bar{u}(\xi = 0,s) = s^{-1} \quad . \tag{21}$$

Since $u(\xi,s)$ must be finite for $\xi \to \infty$, the solution of Eqs. (20) - (21) is

$$\bar{u}(\xi,s) = s^{-1}e^{-(s^2 + s)^{\frac{1}{2}}\xi} \quad . \tag{22}$$

The inverse Laplace transform gives for the velocity field the complex integral

$$u(\xi,t) = (1/2\pi i)\int_{\gamma-i\infty}^{\gamma+i\infty} s^{-1}e^{-(s^2 + s)^{\frac{1}{2}}\xi}e^{st}ds \,. \tag{23}$$

Hence,

$$u(\xi,t) = \partial\Phi(\xi,t)/\partial\xi \tag{24}$$

where

$$\Phi(\xi,t) = -\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} s^{-1}\,\bar{f}(\xi,s)e^{st}ds \tag{25}$$

and

$$\bar{f}(\xi,s) \equiv e^{-(s^2+s)^{\frac{1}{2}}\xi}/(s^2 + s)^{\frac{1}{2}} \quad . \tag{26}$$

According to a known inversion integral,[6)] the inverse transform of Eq. (26) is

$$f(\xi,t) = L^{-1}[\bar{f}(\xi,s)] = e^{-\frac{1}{2}t}\, I_0(\tfrac{1}{2}(t^2 - \xi^2)^{\frac{1}{2}})\, H(t - \xi) \tag{27}$$

where $I_\nu(t)$ is the modified Bessel function of order ν. By Eq. (25),

$$\begin{aligned}\Phi(\xi,t) &= -L^{-1}[s^{-1}\,\bar{f}(\xi,s)] = -\int_0^t L^{-1}[\bar{f}(\xi,s)]dt \\ &= -\int_0^t f(\xi,t)dt \quad ,\end{aligned} \tag{28}$$

i.e.

$$\Phi(\xi,t) = -H(t - \xi)\int_\xi^t e^{-\frac{1}{2}\alpha}I_0(\tfrac{1}{2}(\alpha^2 - \xi^2)^{\frac{1}{2}})d\alpha \quad . \tag{29}$$

From this potential, the dimensionless velocity field is obtained as

$$u(\xi,t) = H(t - \xi)\,[e^{-\frac{1}{2}\xi} + \tfrac{1}{2}\xi\int_\xi^t e^{-\frac{1}{2}\alpha}(\alpha^2 - \xi^2)^{-\frac{1}{2}}I_1(\tfrac{1}{2}(\alpha^2 - \xi^2)^{\frac{1}{2}})d\alpha] \tag{30}$$

in accordance with Eq. (24). By Eq. (17), the corresponding dimensional solution for the velocity field is

$$v(x,t) = \hat{v}H(ct - x)\ \{e^{-x/2c\tau} + (x/2c\tau)\int_{x/c\tau}^{t/\tau} e^{-\frac{1}{2}\alpha}\ [\alpha^2 - (x/c\tau)^2]^{-\frac{1}{2}}\ I_1(\tfrac{1}{2}[\alpha^2 - (x/c\tau)^2]^{\frac{1}{2}})d\alpha\}. \tag{31}$$

Eq. (31) indicates that the transverse stress relaxation wave is discontinuous at $x = ct$, the position of the wave front at time t. At any time $0 \le t \le \infty$, only the region $0 \le x \le ct$ of the fluid is excited by the wave, since $v(x < ct, t) > 0$ and $v(x > ct, t) = 0$ by Eq. (31). The velocity signal $v(0,t) = \hat{v}\ H(t)$ generated at the boundary $x = 0$ at time t is thus transported with finite speed $c = (\mu\ /\rho_o\tau)^{\frac{1}{2}} < \infty$ in form of a discontinuous wave into the fluid space $x > 0$ as $t > 0$ increases.

Application of the asymptotic formula $I_o(z) \sim e^z/(2\pi z)^{\frac{1}{2}}$, $|z| >> 1$, and expansion of $z = \frac{1}{2}[\alpha^2 - (x/c\tau)^2]^{\frac{1}{2}}$ for large α-values in Eq. (31) yields, in the limit $\tau \to 0$, $c\tau \to 0$:

$$v(x,t) = \hat{v}(2/\sqrt{\pi})\int_{\eta}^{\infty} e^{-\beta^2}\ d\beta, \quad \eta \equiv x/2(\frac{\mu}{\rho_o}\ t)^{\frac{1}{2}} \quad . \tag{32}$$

This is the familiar solution of the parabolic Navier-Stokes equations due to Stokes.[7] Eq. (32) suggests that $v(x,t) > 0$ throughout the entire fluid $0 \le x \le \infty$ for any, no matter how small time $t > 0$. Thus, the parabolic Stokes solution gives a completely misleading picture for a shear wave in form of a diffusion process which spreads with infinite speed.

Fig. 1 shows $u(\xi,t)$ versus ξ for $t = 10^0, 10^1, 10^2$, with wave fronts at $\hat{\xi} = 10^0, 10^1, 10^2$. It is seen how the perturbation $u(0,t) = H(t)$ produced at the wall $\xi = 0$ moves in form of a discontinuous wave into the fluid space $\xi \ge 0$ so that an increasing but finite region $0 \le \xi \le \hat{\xi}$ of the fluid is set into motion with increasing t. In the limit $t = \infty$, $u(\xi,t) = 1$ throughout the fluid, $0 \le \xi \le \infty$.

INITIAL-BOUNDARY-VALUE PROBLEM FOR $\vec{v}(x,0)$-PULSE

Another fundamental method for shear wave generation makes use of a velocity pulse $\vec{v}_o = v(x,0)\vec{e}_y$ generated of time t = 0 within a limited region $|x| < \Delta x$. The decay of this velocity pulse occurs in form of a shear wave with velocity field $\vec{v} = \{0, v(x,t), 0\}$ in the y-direction propagating in the x-directions. Accordingly, $\nabla\cdot\vec{v} = \partial v/\partial y = 0$, $v\cdot\nabla\vec{v} = \vec{0}$, and $\vec{v}\cdot\nabla\overleftrightarrow{\Pi} = v\partial\Pi/\partial y = \vec{0}$, since $\overleftrightarrow{\Pi}$ has only a single component $\Pi_{xy} = \Pi(x,t)$. Again, the stransverse wave "behaves" incompressible and linear, and $\nabla p = \vec{0}$ by Eq. (1).

As in the previous problem, Eqs. (1) - (3) give the wave Eqs. (10) and (11) for $\Pi(x,t)$ and $v(x,t)$, respectively. Hence, the shear wave produced by the velocity pulse $\vec{v}_o = v(x,0)\vec{e}_y$ is described by the initial-boundary-value problem:

$$\frac{\partial^2 v}{\partial t^2} + \frac{1}{\tau}\frac{\partial v}{\partial t} = c^2\frac{\partial^2 v}{\partial x^2} \quad , \tag{33}$$

$$v(x,t = 0) = v_o(x) \quad , \quad |x| \leq \infty \quad , \tag{35}$$

$$\partial v(x,t = 0)/\partial t = w_o(x) \quad , \quad |x| \leq \infty \quad , \tag{35}$$

where $|w_o(x)| \geq 0$ is included for reasons of generality. The solution of Eqs. (33) - (35) is accomplished by means of Riemann's method,[8)]

$$v(x,t) = e^{-t/2\tau}\{\tfrac{1}{2}[v_o(x-ct) + v_o(x+ct)] + \tfrac{1}{2}\int_{x-ct}^{x+ct}\Psi(x,t,\alpha)d\alpha\} \tag{36}$$

where

$$\Psi(x,t,\alpha) = v_o(\alpha)(t/2\tau)\ I_1(\frac{1}{2c\tau}[c^2t^2 - (\alpha - x)^2]^{\frac{1}{2}})/[c^2t^2 - (\alpha - x)^2]^{\frac{1}{2}}$$

$$+ \frac{1}{c}[w_o(\alpha) + \frac{1}{2\tau}v_o(\alpha)]\ I_o(\frac{1}{2c\tau}[c^2t^2 - (\alpha - x)^2]^{\frac{1}{2}}) \quad . \tag{37}$$

As a concrete example for the initial conditions in Eqs. (34) - (35), an initial velocity distribution of the form of a Dirac pulse is chosen,

$$v_o(x) = \hat{v}_o\delta(x), \quad w_o(x) = 0, \quad |x| \leq \infty \quad . \tag{38}$$

In this case, the general solution in Eqs. (36) - (37) becomes, in dimensionless form,

$$u(\xi,t) = \tfrac{1}{2}\, e^{-t}\, [\delta(\xi - t) + \delta(\xi + t)$$

$$+ I_1([t^2 - \xi^2]^{\frac{1}{2}})/[t^2 - \xi^2]^{\frac{1}{2}} + I_o([t^2 - \xi^2]^{\frac{1}{2}})], \quad |\xi| \leq t,$$

$$= 0 \qquad , \quad |\xi| > t, \tag{39}$$

where

$$u(\xi,t) = v(x,t)/(\hat{v}_o/2c\tau), \quad \xi = x/2c\tau, \quad t = t/2\tau \quad . \tag{40}$$

Eq. (39) indicates that the shear wave spreads in the space $|\xi| \leq \infty$ in form of a symmetrical wave, $u(-\xi,t) = u(+\xi,t)$ due to the symmetry of the initial conditions (38). The wave is discontinuous at its fronts $\hat{\xi} = \pm t$, which propagate with the speed

$$v(\hat{x},t) = \pm c, \quad \acute{x} = \pm ct \quad . \tag{41}$$

In the limit $\tau \to 0$, $c\tau \to 0$, application of the asymptotic formula $I_\nu(z) \sim e^z/(2\pi z)^{\frac{1}{2}}$, $|z| >> 1$, and expansion of $z = [\alpha^2 - \xi^2]^{\frac{1}{2}}$ for large α-values in Eq. (39) yields

$$u(\xi,t) = (2\pi t)^{-\frac{1}{2}}\, e^{-\xi^2/2t}, \quad |\xi| \leq \infty \quad . \tag{42}$$

This is the corresponding solution of the parabolic Navier-Stokes equations. Eq. (42) would indicate that the shear wave has the form of a Gaussian extending from $\xi = -\infty$ to $\xi = +\infty$ for any, no matter how small time $t > 0$ (corresponding to an infinite speed of propagation). It is obvious that the solution (42) is physically not meaningful.

In Fig. 2, the dimensionless velocity field $u(\xi,t)$ of the shear wave is shown versus ξ for $t = 10^0, 10^1, 10^2$, the wave fronts being in each case at $\hat{\xi} = \pm t$. Due to the finite wave speed c, the fluid is not excited in the region $|\xi| > t$ ahead of the wave fronts. The shape of the wave is flat with relatively steep flanks leading to the discontinuous fronts. Thus, the shear wave does not resemble the Gaussian of the parabolic theory, Eq. (40).

CONCLUSIONS

A generalization of the Navier-Stokes equations is presented considering viscous stress relaxation, which results in a physically meaningful theory for transverse waves in viscous fluids. The fundamental speed of the stress relaxation waves is given by $c = (\mu/\rho\tau)^{\frac{1}{2}}$, where μ is the viscosity, ρ is the density, and τ is the relaxation time of the stress tensor. For any medium,[9] it is $c \lesssim c_s$, where c_s is the speed of the longitudinal waves, e.g., $c = 1.3 \times 10^5$ cm sec^{-1} and $c_s = 1.5 \times 10^5$ cm sec^{-1} for water at T = 20°C and p_o = 1 atm.

Exact solutions are derived for stress relaxation waves propagating in the x-direction due to velocity pulses $\vec{v}(0,t)$ and $\vec{v}(x,0)$ in the y-direction, respectively. For the geometry of these transverse waves, the nonlinear, generalized Navier-Stokes equations become linear, so that the solutions given hold for waves of arbitrary intensity. The solutions are discontinuous at the wave fronts, which is typical for hyperbolic field equations. The corresponding solutions of the conventional Navier-Stokes equations indicate a diffusion process with infinite wave speed, i.e., give a qualitatively and quantitatively insufficient picture of the propagation of transverse waves in fluids.

In the simplified stress relaxation equation (3) proposed, the term $\overleftrightarrow{\Pi}\cdot\nabla\vec{v}$ is neglected since it is of the order-of-magnitude of $(\mu/\tau)|\nabla\vec{v}|^2$, which is nonlinear in the derivatives. It should be noted that the term $\overleftrightarrow{\Pi}\cdot\nabla\vec{v}$ vanishes exactly for the wave problems treated above, $\overleftrightarrow{\Pi}\cdot\nabla\vec{v}=\vec{0}$ since $\vec{v} = \{0,v(x,t),0\}$ and $\overleftrightarrow{\Pi}$ has only a single component $\Pi_{xy}(x,t)$. For this reason, the solutions presented are exact solutions of the nonlinear Navier-Stokes equations with viscous stress relaxation.

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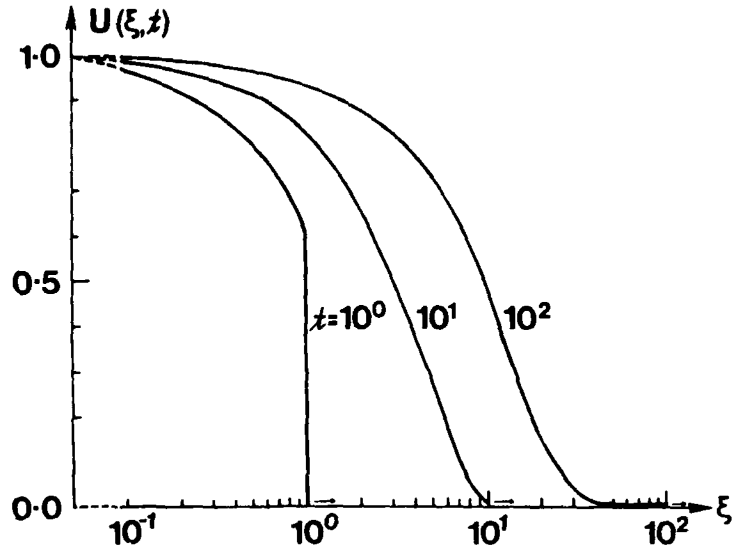

Fig. 1: $u(\xi,t)$ versus ξ for $t = 10^0$, 10^1, 10^2, with $u(0,t) = H(t)$.

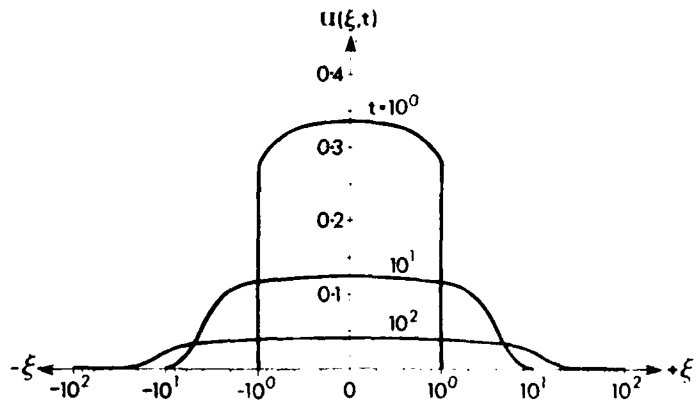

Fig. 2: $u(\xi,t)$ versus ξ for $t = 10^0, 10^1, 10^2$, with $u(\xi,0) = \delta(\xi)$.

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The report contains (1) a dimensional analysis of the possible electrical conductivities of nonideal classical and quantum plasmas, (2) a kinetic theory of the electrical conductivities of nonideal plasmas with I) Maxwell and II) Fermi distribution of the electrons, based on an effective, shielded Coulomb potential for conditions of intermediate nonideality, and (3) a statistical theory of the free energy of nonideal plasmas considering quasi-static Coulomb interactions (Madelung energy) and dynamic Coulomb interactions (low and high frequency plasma oscillations). Furthermore, (4) a Hamilton theory is developed for many-component plasmas, which forms the basis for a statistical thermodynamics of nonideal plasmas with longitudinal field interactions. This formalism is applied (5) to the statistical determination of the distribution function of turbulent velocity flucuations. In the Appendix, (6) an unrelated subject on pulsed stress relaxation waves is discussed for signal transmission and system detection in water.

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